Identifying coloring of graphs

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Proper coloring

Two adjacent vertices have distinct colors.

\[ B_t(u) = \{ v \mid d(u, v) \leq t \} \]

For any edge \( uv \), \( c(B_0(u)) \neq c(B_0(v)) \)
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For any edge $uv$, $c(B_0(u)) \neq c(B_0(v))$
Locally identifying coloring (lid-coloring)

Two adjacent vertices have distinct colors in their neighborhood.

For any edge $uv$, $c(B_0(u)) \neq c(B_0(v))$ and $c(B_1(u)) \neq c(B_1(v))$.
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\( \chi_{lid}(G) : \) lid-chromatic number
An example: the path

With 4 colors:

```
χ_{lid}(P_k) ≤ 4
```

Is it possible with 3 colors?

```
χ_{lid}(P_k) = 3 ⇔ k is odd
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An example: the path

With 4 colors:

1 2 3 4 1 2 3 4

So:

\[ \chi_{\text{lid}}(P_k) \leq 4 \]

Is it possible with 3 colors?

1 2 1 2 3

\[ \chi_{\text{lid}}(P_k) = 3 \iff k \text{ is odd} \]
An example: the path

With 4 colors:

1, 2, 3, 4

1, 2
1, 2, 3
2, 3, 4
1, 3, 4
1, 2, 4
1, 2, 3
2, 3, 4
3, 4

So:

\[ \chi_{lid}(P_k) \leq 4 \]
An example: the path

With 4 colors:

1, 2, 1, 2, 3, 2, 3, 4, 1, 3, 4, 1, 2, 4, 1, 2, 3, 2, 3, 4, 3, 4

So:

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Is it possible with 3 colors?
An example: the path

With 4 colors:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1, 2 & 1, 2, 3 & 2, 3, 4 & 1, 3, 4 & 1, 2, 4 & 1, 2, 3 & 2, 3, 4 & 3, 4 \\
\end{array}
\]

So:

\[
\chi_{lid}(P_k) \leq 4
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Is it possible with 3 colors?

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 2 & 1 & 2 & 3 & 2 \\
1, 2 & 1, 2, 3 & 2, 3 & 1, 2, 3 & 1, 2 & 1, 2, 3 & 2, 3 & 2, 3 \\
\end{array}
\]
An example: the path

With 4 colors:

1, 2, 1, 2, 3, 2, 3, 4, 1, 3, 4, 1, 2, 4, 1, 2, 3, 2, 3, 4, 3, 4

So:

$$\chi_{lid}(P_k) \leq 4$$

Is it possible with 3 colors?

1, 2, 1, 2, 3, 2, 3, 1, 2, 3, 1, 2, 1, 2, 3, 2, 3, 2, 3

$$\chi_{lid}(P_k) = 3 \iff k \text{ is odd}$$
Related works

With edge colorings:
- Vertex-distinguishing edge colorings (Observability of a graph) (Hornak et al, 95’),
- Adjacent vertex-distinguishing edge colorings (Zhang et al, 02’)

With total colorings:
- Adjacent vertex-distinguishing total colorings (Zhang, 05’)

Link with chromatic number

Do we need much more than $\chi(G)$ colors?
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An example with $\chi(G) = 3$ and $\chi_{lid}(G) \geq k$
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What about “good classes” for classical colorings?
Bipartite graphs

- $L_0$
- $L_1$
- $L_2$
- $L_3$
- $L_4$
Bipartite graphs

\[ L_0 \rightarrow 1 \quad 1, 2 \]
\[ L_1 \rightarrow 2 \quad 1, 2, 3 \]
\[ L_2 \rightarrow 3 \quad 2, 3, 4 \text{ or } 2, 3 \]
\[ L_3 \rightarrow 4 \quad 1, 3, 4 \text{ or } 3, 4 \]
\[ L_4 \rightarrow 1 \quad 1, 4 \]
Bipartite graphs

General bounds: $3 \leq \chi_{lid}(B) \leq 4$
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$\chi_{lid}(B) = 3$: 

![Diagram of a bipartite graph with 3 colors]

In general, 3-Lid-Coloring is NP-complete in bipartite graphs.
Bipartite graphs

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← ? →

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Link with 2-coloring of hypergraph
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1, 2, 3

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- 3-LID-COLORING in bipartite graph is NP-Complete
- Polynomial if $B$ regular, if $B$ is planar with maximum degree 3.
To perfect graph: $k$-trees

Lid-coloring of 2-trees with 6 colors:

- Color the triangle with colors 1, 2, 3
- Step:

![Diagram of a triangle with vertices labeled 1, 2, 3 and a line segment from vertex 1 to vertex i.]
To perfect graph : \textit{k-graphs}

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- Step:

\[ i + 3[6] \]

- We always have:
  - proper coloring
  - no edge \((i, i + 3)\)
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  $i$

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To perfect graph: $k$-trees

We can extend the construction to $k$-trees:

→ A $k$-tree has lid-chromatic number at most $2k + 2$

This bound is sharp: $P_{2k+2}^k$
Perfect graphs

- Bipartite graphs: $4 = 2\omega$
- $k$-trees: $2k + 2 = 2\omega$, 
Perfect graphs

- Bipartite graphs: $4 = 2\omega$
- $k$-trees: $2k + 2 = 2\omega$,
- Split graphs: $2\omega - 1$
- Cographs: $2\omega - 1$
- ...
Perfect graphs

- Bipartite graphs: \( 4 = 2\omega \)
- \( k \)-trees: \( 2k + 2 = 2\omega \),
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**Question:** Can we color any perfect graph \( G \) with \( 2\omega(G) \) colors?
Perfect graphs

- Bipartite graphs: $4 = 2\omega$
- $k$-trees: $2k + 2 = 2\omega$,
- Split graphs: $2\omega - 1$
- Cographs: $2\omega - 1$
- ...

**Question:** Can we color any perfect graph $G$ with $2\omega(G)$ colors?

**No!**

$V_1, V_2, V_3$ stable sets of size $l$
Planar graphs:

- No general bound, but...
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- Examples with at most 8 colors
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Outerplanar graphs:

- General bound: 20 colors,
- Max outerplanar graphs: 6 colors,
- Without triangles: 8 colors,
Planar graphs:
- No general bound, but...
- With large girth (36): 5 colors,
- Examples with at most 8 colors

Outerplanar graphs:
- General bound: 20 colors,
- Max outerplanar graphs: 6 colors,
- Without triangles: 8 colors,
- Examples with at most 6 colors
A bound for outerplanar graphs

- a layer = union of paths,
- 5 colors in a layer,
- $4 \times 5 = 20$
Some open questions

- Is $\chi_{lid}$ bounded for planar graphs?
- For which graphs $\chi_{lid} = \chi$?
- Link with maximum degree $\Delta$?
- What about a global version?
Thanks !