Distance edge coloring and collision-free communication in wireless sensor networks

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Abstract

Motivated by the problem of link scheduling in wireless sensor networks where different sensors have different transmission and interference ranges and may be mobile, we study the problem of *distance edge coloring* of graphs which is a generalization of proper edge coloring. Let G be a graph modeling a sensor network. An ℓ-distance edge coloring of G is a coloring of the edges of G such that any two edges within distance ℓ of each other are assigned different colors. The parameter ℓ is chosen so that the links corresponding to two edges that are assigned the same color do not interfere. We investigate the ℓ-distance edge coloring problem on several families of graphs which can be used as topologies in sensor deployment. We focus on determining the minimum number of colors needed and on optimal coloring algorithms.

Keywords: *link scheduling, sensor networks, distance edge coloring, grids, hypercubes, power graphs*

1 Introduction

Wireless sensor networks are widely used in a variety of control applications in health and military domains. A sensor appears as a miniaturized device with sensing, processing and wireless communication capabilities. Depending on the application, sensors are deployed according to either a carefully predefined topology or a random topology and form a wireless network. Once deployed, sensors are intended to collect data and transmit them to some base station or processing center. Transmission of collected data is generally achieved using multihop communications.

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In fact, sensors are not always able to reach the base station with a direct transmission. Their transmission power is low obliging them to use other sensors to relay messages to the base station. So, when a sensor needs to communicate its collected data to the base station, it sends it to one of its neighbors. The neighbor that receives the message transmits it to another neighbor according to some pre-computed route. It follows that multiple sensors may be forwarding data to their neighbors at the same time increasing the probability of message collision [24]. Because sensors are battery powered, energy conservation is a crucial issue. For this reason, message collision is undesirable in such networks. When a collision occurs, the involved messages are irrecoverable and the power needed to transmit them is wasted. So, collision-free communication is a real challenge in sensor networks. To avoid collision, transmission scheduling is needed. *TDMA* (Time Division Multiple Access) link scheduling is generally used in sensor networks [10]. Time division multiplexing is the problem of assigning timeslots to sensors. Two sensors u and v can transmit in the same timeslot if and only if u does not interfere with the communication of v, and v does not interfere with the communication of u. Interferences occur among sensors due to the broadcast nature of the wireless medium. There are two types of interferences: primary interferences and secondary interferences [30]. A primary interference occurs when a sensor is involved in more than one communication task during a timeslot. A secondary interference occurs when a sensor is in the transmission range of a communication between other nodes [30]. Secondary interferences may induce what are commonly known as the hidden terminal problem and the exposed terminal problem [5]. Figure 1 illustrates these two problems.

TDMA MAC protocols attempt to minimize the number of timeslots assigned to links while guaranteeing an interference-free schedule. This problem was shown to be NP-complete [1].

The timeslot assignment problem in wireless networks is associated with the problem of edge coloring in the graph that represents the network [3, 14, 29]. The vertices of the graph are the networked devices and the edges are communication links. Given a graph $G = (V(G), E(G))$ where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, a proper edge coloring of G consists of assigning colors to the edges of G such that no two adjacent edges have the same

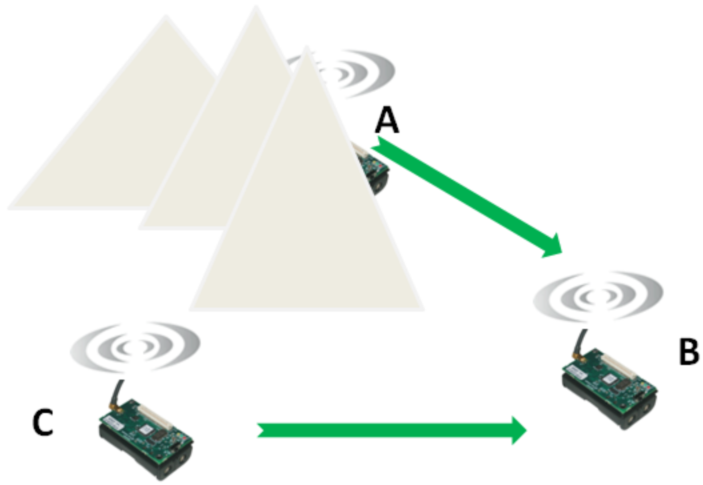

Figure 1: Sensor B can hear both A and C, but A and C cannot hear each other. "A hidden terminal scenario results when C attempts to transmit while A is transmitting to B. An exposed terminal scenario results if B is transmitting to A when C attempts to transmit" [5].

color [4, 6]. The problem of timeslot scheduling consists then of assigning two timeslots to each color, if we consider that each edge in the graph corresponds to two links: a link for each communication direction. Vizing [33] proved that a proper edge coloring of a graph can be obtained using either $\triangle$ or $(\triangle + 1)$ colors, where $\triangle$ denotes the maximum degree of a vertex in the graph. However, there are cases where proper edge coloring is insufficient to solve the timeslot assignment problem, mainly in the case of the hidden terminal problem [5]. In this case, *strong edge coloring* is used [15, 27]. This consists of assigning different colors to edges incident to the same vertex or to the neighbors of the same vertex. However this solution results in the exposed terminal problem [5] and is not necessarily the best [14]. In this work, we consider a more general modeling of this problem that takes into account more specific characteristics of wireless sensor networks and especially the two following ones:

- Sensors may be heterogeneous and have different transmission capabilities. We suppose that each sensor v_i has a transmission range r_i and an interference range R_i, where $R_i > r_i$. So, to avoid interferences even strong edge coloring of the communication graph is not sufficient.

- The topology of a sensor network is dynamic. Sensors may be mobile and even if they are static, they can turn on their transceivers during the assigned timeslots and turn off their transceivers when not transmitting or receiving. This helps considerably in preserving energy. To deal with a dynamic topology, the common solution is either to update the timeslot assignment dynamically or to rely on self-stabilizing solutions [18]. However this is not appropriate [28] for resource-constrained environments such as sensor networks. Indeed, energy preservation is ranked as one of the most important properties of MAC protocols for WSNs well ahead of channel utilization.

To deal with these restrictions and mainly the one related to energy consumption induced by a dynamic update of the schedule, we propose using a generalization of strong edge coloring called *ℓ-distance edge coloring* where an integer ℓ bounds the distance between two edges that are assigned the same color, i.e., timeslot. For an integer $\ell \geq 0$, an *ℓ-distance edge coloring* of G colors all edges of G so that any two edges e and e' with $dist(e, e') \leq \ell$ have different colors. The distance between two edges $e = (u, v)$ and $e' = (u', v')$ of G is defined as $\min\{dist(u, u'), dist(u, v'), dist(v, u'), dist(v, v')\}$ where the distance $dist(u, v)$ between two vertices u and v is the number of edges in the shortest path between u and v in the graph G. A 0-distance edge coloring is the ordinary proper edge coloring. The minimum number of colors used to color a graph with an ℓ-distance edge coloring is called the *ℓ-chromatic index* and is denoted by $\chi'_\ell(G)$.

In this paper, we focus on some classes of graphs that can be used as topologies in sensor networks and study the feasibility of an ℓ-distance edge coloring. For link scheduling purposes, we are interested in determining the minimum number of colors and in coloring algorithms. To simplify the study of different classes of graphs, we rely on a unified framework based on the structure of ℓ-balls defined in Section 3. The paper is organized as follows. In Section 2, we review related graph colorings. In Section 3, we give the exact values of the ℓ-chromatic index for hypercubes and meshes, and we propose coloring algorithms for these graphs. In Section 4, we consider this coloring parameter for power graphs of trees and cycles. Section 5 gives our concluding remarks.

2 Related Work

The timeslot assignment problem was studied as a graph coloring problem in several wireless networking conditions [1, 3, 17]. Detailed surveys can be found in [2, 9]. For sensor networks, several proposals for timeslot assignment algorithms are based on proper edge coloring [10, 14] and strong edge coloring [18, 24, 26, 28] of graphs. Using ℓ-distance edge coloring for timeslot assignment in wireless sensor networks has already been proposed in [10] where the authors studied proper edge coloring in directed graphs. Distance edge coloring was first studied by Skupien in [32] where he gives exact values of the ℓ-chomatic index of paths, cycles and trees. In the same paper, Skupien studied the ℓ-chromatic index of hypercubes using coding theory. He proposed bounds as a function of the minimum length of the binary code labeling the vertices of the hypercube. In [20], Jendrol and Skupien also studied the ℓ-chomatic index of $\triangle$-regular planar graphs and proposed several bounds for different values of $\triangle$ and ℓ. Recently, Ito *et al.* [19] studied the decision problem of whether a graph has an ℓ-distance edge coloring with a given number α of colors using dynamic programming. They proposed an algorithm that determines in time $O(n(\alpha+1)^{2^{2(k+1)(\ell+1)+1}})$ whether a partial k-tree G has an ℓ-distance edge coloring with a given number α of colors, where n is the number of vertices in G. Then, using a binary search technique, one can compute the ℓ-chromatic index of G by applying the proposed algorithm for at most $\log_2 m$ values of α, where m is the number of edges in G. According to the authors, this algorithm can be easily modified so that it finds a coloring. The authors also proposed, in the same paper, a 2-approximation polynomial-time algorithm to find an ℓ-distance edge coloring of a given planar graph G with at most $2\chi'_\ell(G)$ colors. In [22], the authors study the ℓ-distance edge coloring for sparse random graphs. In [13], the authors studied the ℓ-chromatic index of the Kronecker product of two paths, two cycles, a path by a cycle, two stars and the graph K_2 by other graphs. In [21], the authors prove two upper bounds for the ℓ-chromatic index. One is a bound of $(2-\varepsilon)\Delta^\ell$ for graphs of maximum degree at most Δ, where ε is some absolute positive constant independent of ℓ. The other is a bound of $O(\frac{\Delta^{\ell}}{\log \Delta})$ (as $\Delta \to \infty$) for graphs of maximum degree at most Δ and girth at least $2\ell + 1$.

We can observe that an ℓ-distance edge coloring cannot be deduced from another existing coloring. Of course, coloring edges of a graph subject to distance conditions is equivalent to coloring the vertices of the corresponding line graph subject to distance conditions. When the timeslot or channel assignment problem is evoked, many colorings are distinguished (on vertices or edges) with close definitions to ℓ-distance edge coloring. In this context, we can cite the $L(h,k)$*-coloring* introduced in [16] and defined as follows: for nonnegative integers h and k, an $L(h,k)$*-coloring* of a graph G is an assignment of colors to the vertices of G such that if u and v are adjacent in G, then $|c(u)-c(v)| \geq h$ while if $dist(u,v) = 2$ we have $|c(u)-c(v)| \geq k$. However, no condition is imposed on colors assigned to u and v if $dist(u,v) \geq 3$. In particular, the case $h = 2$ and $k = 1$ (i.e., $L(2,1)$*-coloring* also called $L(2,1)$*-labeling*) is considered in [7, 11, 31]. The $L(h,k)$-coloring has been generalized as follows: for nonnegative integers $d_1, d_2, \ldots, d_k$, an $L(d_1, d_2, \ldots, d_k)$*-coloring* of a graph G is an assignment of colors to the vertices of G such that $|c(u)-c(v)| \geq d_i$ whenever $dist(u,v) = i$ for $1 \leq i \leq k$. A *k-radio coloring* is defined [8,25,35] as an assignment of colors to the vertices of G such that, for an integer k with $1 \leq k \leq Diameter(G)$, two distinct vertices u and v satisfy $dist(u,v) + |(c(u) - c(v)| \geq k + 1$.

In these colorings, constraints appear for any distance between the vertices. In our coloring, only the distance ℓ is constrained which makes such a coloring independent from the above colorings. Moreover, since the constraint of the ℓ-distance edge coloring is based on the distance between two edges with the same color in a graph, it is natural to think that the problem can be reduced to the determination of the chromatic index for its power graph (for a natural number p, the p^{th} power graph G^p is the graph obtained from G by adding an edge between every pair of vertices at distance p or less). However, if we consider the path P_8 of order 8, for $\ell = 3$, we have $\chi'_3(P_8) = \ell + 2 = 5$ [12], while the chromatic index of its ℓ^{th} power graph is $\chi'(P_8^3) = 2\ell = 6$ (by Vizing's theorem [33]).

In this paper, we consider ℓ-distance edge coloring for meshes, hypercubes and power graphs of trees and cycles. Note that the treewidths of these graphs are not bounded by a constant, and hence the algorithm of [19] does not work efficiently for those graphs.

3 Distance edge coloring of hypercubes and meshes

We use the structure of an ℓ-ball defined below as a common framework to study different classes of sensor networking topologies.

Definition 1. *For a graph G and an integer $\ell \geq 0$, we define an ℓ-ball as a maximum subgraph $S_\ell \subseteq G$ such that every two edges of S_ℓ are at distance ℓ or less from each other.*

For an ℓ-distance edge coloring, it is clear that all the edges of S_ℓ must have pairwise distinct colors. For each considered class of graphs, we first show how to build an ℓ-ball S_ℓ. In general, it is convenient to start building S_ℓ at the middle of S_ℓ and move outwards. Depending on the target graph, this middle may be a vertex (the center) or two vertices joined by an edge (the bicenter). In the following we denote the middle of S_ℓ by $\mathcal{C}_\ell$ be it either a center or a bicenter. Thus, we will first determine $\mathcal{C}_\ell$ and then build an ℓ-ball S_ℓ. We note also that the middle may not be unique and consequently the ℓ-ball S_ℓ is also not unique. In this case, we will distinguish the different ℓ-balls by their middles and denote by $S_\ell^{\mathcal{C}}$, the ℓ-ball of middle $\mathcal{C}$ when needed. In most cases, an ℓ-ball S_ℓ is simply the subgraph induced by the set of vertices that are at distance less than or equal to $\left\lfloor \frac{\ell}{2} \right\rfloor + 1$ from $\mathcal{C}_\ell$, i.e., from the vertices of $\mathcal{C}_\ell$. If $Q = \cup_{x \in \mathcal{C}_\ell} \{u \in V(G), dist(u,x) \leq \left\lfloor \frac{\ell}{2} \right\rfloor + 1\}$, the ℓ-ball S_ℓ is $S_\ell = G[Q]$. In some cases however and particularly when $\mathcal{C}_\ell$ is a bicenter, S_ℓ needs a particular construction. If there is no ambiguity, the ℓ-ball $S_\ell(G)$ will be denoted S_ℓ. Once an ℓ-ball S_ℓ is built for a graph G, we can compute a lower bound for the ℓ-chromatic index of G: it is the number of edges of S_ℓ. To determine an upper bound for the ℓ-chromatic index of G, we use a coloring algorithm.

We first study the ℓ-chromatic index of some simple graphs, namely hypercubes and meshes, and then we consider this parameter for power graphs of trees and cycles.

3.1 Hypercubes

A d-dimensional hypercube or d-cube H_d is a graph, $H_d = (V(H_d), E(H_d))$, defined recursively as of the cartesian product[1] of two graphs as follows: $H_1 = K_2$ and $H_d = K_2 \times H_{d-1}$. Additionally, H_d may be constructed using two $(d-1)$-cubes H^1_{d-1} and H^2_{d-1} and joining equivalent edges together. The joining edges are called the (d^{th}) dimension of H_d. In the following we prove an exact value for the ℓ-chromatic index of hypercubes and we propose a coloring algorithm. We first construct an ℓ-ball for the hypercube H_d.

Definition 2. *Let $\ell \geq 0$, $d \geq \ell+1$ and let H_d be a d-dimensional hypercube. An ℓ-ball $S_{\ell,d} \subseteq H_d$ is defined as the union of two graphs G_1 and G_2, $S_{\ell,d} = G_1 \cup G_2$, where G_1 is an $(\ell+1)$-cube contained in one of the $(d-1)$-cubes that compose H_d and G_2 is an edge-induced subgraph on the set of edges incident to exactly one of the ℓ-cubes that compose G_1.*

Figure 2 shows examples of the construction of some ℓ-balls. For instance, for H_3 and $\ell = 1$, $S_{1,3} = G_1 \cup G_2$ is such that G_1 is a square (i.e., a 2-cube) and G_2 is the subgraph induced by the two edges incident to one of the 1-cubes (i.e., line segments) of G_1.

The following lemma gives the size of an ℓ-ball $S_{\ell,d}$ of H_d:

Lemma 1. *Let $\ell \geq 0$, $d \geq \ell+1$, H_d be a hypercube and $S_{\ell,d} \subseteq H_d$ be an ℓ-ball of H_d. Then $|E(S_{\ell,d})| = d2^\ell$ and for any $e, e' \in E(S_{\ell,d})$, $dist(e, e') \leq \ell$.*

Proof. According to Definition 2, the number of edges of $S_{\ell,d}$ is equal to the number of edges of $H_{\ell+1}$ plus the number of edges of G_2. $H_{\ell+1}$ has $(\ell+1)2^\ell$ edges and G_2 contains $(d-\ell-1)\,2^\ell$ edges. So, $|E(S_{\ell,d})| = (\ell+1)\,2^\ell + (d-\ell-1)\,2^\ell = d2^\ell$.

We have for any $e, e' \in E(H_{\ell+1})$, $dist(e, e') \leq \ell$ and one of the endpoint of each edge of G_2 belongs to $V(H_{\ell+1})$. Consequently, for any $e, e' \in E(S_{\ell,d})$, $dist(e, e') \leq \ell$.

□

In the following, we present the main result of this section.

[1] The cartesian product of two graphs G and H is the graph denoted by $G \times H$ whose vertex set is the (ordinary) cartesian product $V(G) \times V(H)$ and such that two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if u is adjacent to u' and v is adjacent to v'.

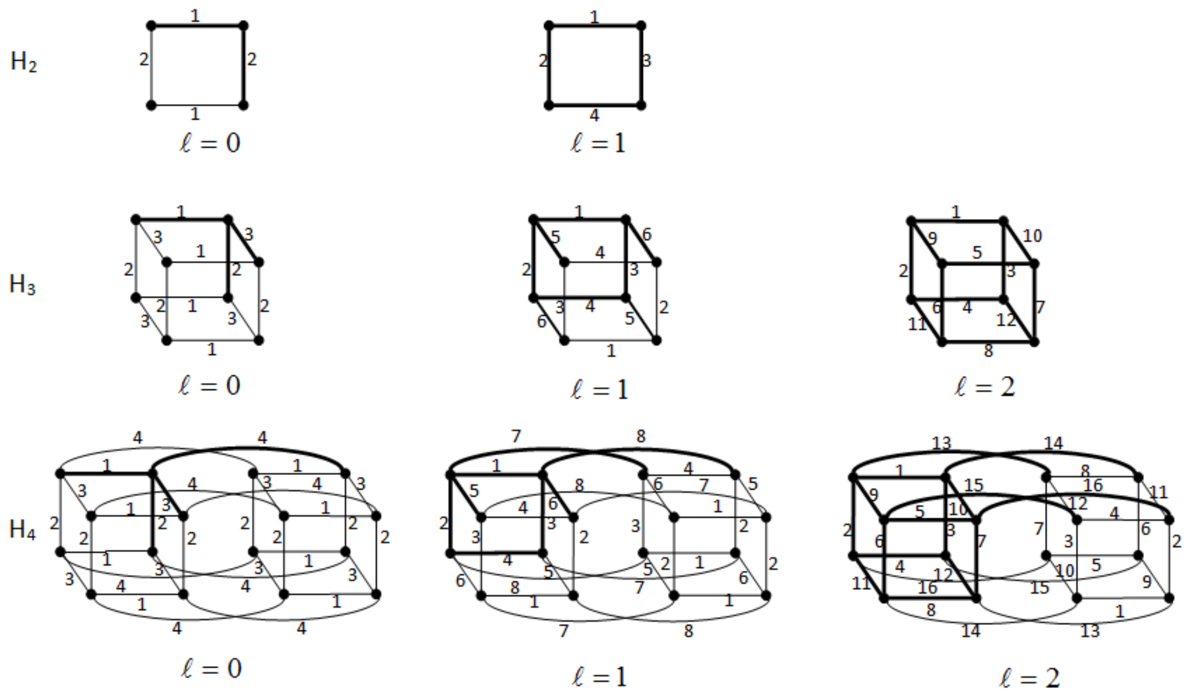

Figure 2: Examples of ℓ-distance edge colorings of H_i, $1 \le i \le 4$. The ℓ-ball $S_{\ell,d}$ is given by the bold edges.

Theorem 1. *Let H_d be a d-dimensional hypercube. Then the ℓ-chromatic index of H_d is given by:*

$$\chi'_\ell(H_d) = \begin{cases} d2^{d-1} & \text{if } \ell \ge d-1 \quad (a) \\ d2^{\ell} & \text{otherwise} \quad (b) \end{cases}$$

Proof. Case (a). Let $e, e' \in E(H_d)$ be two edges of a d-cube H_d. Then the distance between these two edges is at most equal to $d-1$. However, if $\ell \ge d-1$, we have $dist(e, e') \le \ell$. All edges of H_d have different colors and $\chi'_\ell(H_d) = |E(H_d)| = d2^{d-1}$.

Case (b). According to Lemma 1, $\chi'_\ell(H_d) \ge d2^{\ell}$ (i.e., the number of edges in the ℓ-ball $S_{\ell,d}$). To prove the upper bound, we propose a coloring algorithm for H_d that uses $d2^{\ell}$ colors. The algorithm relies on the two following properties of hypercubes:

1. A d-cube H_d is composed of two isomorphic $(d-1)$-cubes H^1_{d-1} and H^2_{d-1} connected by $2^{(d-1)}$ joining edges. We say that H^2_{d-1} is the copy of H^1_{d-1}. By this isomorphism, each edge of H^1_{d-1} has a copy in H^2_{d-1} (i.e., the two edges are connected by a pair of joining

edges).

2. For each edge e in H^1_{d-1} there is a unique edge e' in H^2_{d-1} such that $dist(e,e') = d-1$. We say that e' is the $(d-1)$-image of e in H^2_{d-1}. Thus, if we consider $H_{\ell+2}$ then for each edge e in $H^1_{\ell+1}$ there is a unique edge e' in $H^2_{\ell+1}$ such that $dist(e,e') = \ell+1$. So, e and e' can have the same color in an ℓ-distance edge coloring. Figure 3 illustrates these edges for $\ell = 2$.

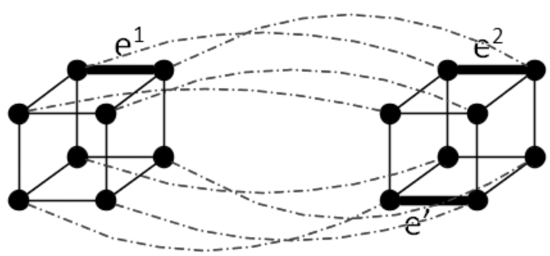

Figure 3: e^2 is the copy of e^1 and e' is its $(\ell+1)$-image. e^1 and e' are at distance $\ell+1$ and can have the same color.

The main idea of our coloring algorithm is the coloring of $H_{\ell+2}$. $H_{\ell+2}$ is composed of two $(\ell+1)$-cubes $H^1_{\ell+1}$ and $H^2_{\ell+1}$ and $2^{\ell+1}$ joining edges. The algorithm proceeds as follows:

- We color $H^1_{\ell+1}$ with $|E(H_{\ell+1})| = (\ell+1)2^{\ell}$ colors (all the edges of $H_{\ell+1}$ are at distance less than or equal to ℓ from each other, and no particular order is required to assign these different colors to $H^1_{\ell+1}$). Then, we use the colors of $H^1_{\ell+1}$ to color $H^2_{\ell+1}$ by using the following procedure that we denote by ColorCopy(): For each non-colored edge e' in $H^2_{\ell+1}$ set $c(e') = c(e)$ where e' is the $(\ell+1)$-image of e.

- To color the $2^{\ell+1}$ joining edges, we introduce 2^{ℓ} new colors and proceed as follows:

  we note, first, that at this step of the coloring of $H_{\ell+2}$, only the $2^{\ell+1}$ joining edges are not yet colored, we will denote them by non-colored edges. According to Figure 4, by reordering the dimensions of $H_{\ell+2}$, we obtain a new view of this hypercube, that we denote by $H'_{\ell+2}$, where the non-colored edges are contained in the two $(\ell+1)$-cubes, $H'^1_{\ell+1}$ and $H'^2_{\ell+1}$, that compose $H'_{\ell+2}$. Note that $H'^1_{\ell+1}$ and $H'^2_{\ell+1}$ are partially colored. 2^ℓ non-colored edges are in $H'^1_{\ell+1}$ and the 2^ℓ other non-colored edges are in $H'^2_{\ell+1}$. We color the 2^ℓ non-colored edges of $H'^1_{\ell+1}$ with the 2^ℓ new colors. Then, we use procedure ColorCopy() to complete the coloring of $H'^2_{\ell+1}$ with the same colors.

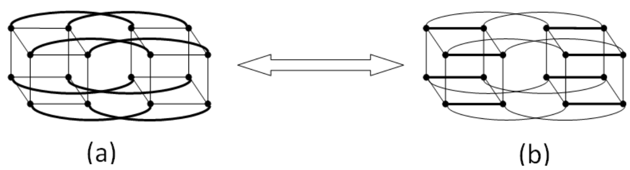

Figure 4: By reordering the dimensions of H_4, the joining edges (bold edges) of the first hypercube (a) become edges of the two 3-cubes that compose the second hypercube (b).

To color H_d, $d > \ell + 2$, the algorithm first colors a subgraph $H_{\ell+1} \subset H_d$ with $|E(H_{\ell+1})| = (\ell + 1)2^\ell$ colors. Then, it consists of a recursive procedure that requires $d - \ell - 1$ iterations. During each iteration, we color the hypercube $H_{\ell+1+i},\ i \in [1, d - \ell - 1]$ as follows:

- $H_{\ell+1+i}$ is composed of $H^1_{\ell+i}$ and $H^2_{\ell+i}$ and $2^{\ell+i}$ joining edges. $H^1_{\ell+i}$ is colored. We color $H^2_{\ell+i}$ with the colors of $H^1_{\ell+i}$ as follows: $H^2_{\ell+i}$ can be viewed as containing 2^{i-1} $(\ell + 1)$-cubes. Figure 5 illustrates an example of this representation of a hypercube. Each $(\ell + 1)$-cube $H^2_{\ell+1}$ of $H^2_{\ell+i}$ has a copy $H^1_{\ell+1}$ in $H^1_{\ell+i}$. We color each $H^2_{\ell+1}$ of $H^2_{\ell+i}$ with the colors of its copy using procedure ColorCopy() as described in the algorithm that colors an $H_{\ell+2}$.

- The $2^{\ell+i}$ joining edges can be viewed as joining $(\ell + 1)$-cubes (see Figure 5). Each $H^1_{\ell+1}$ of $H^1_{\ell+i}$ is connected to its copy $H^2_{\ell+1}$ of $H^2_{\ell+i}$ with $2^{\ell+1}$ edges. As there are 2^{i-1} $(\ell + 1)$-cubes, this gives the $2^{\ell+i}$ total joining edges. So, we introduce 2^ℓ new colors and we use them to color each set of $2^{\ell+1}$ edges that join an $H^1_{\ell+1}$ to an $H^2_{\ell+1}$. We use for this the same method as the one used to color the joining edges in an $H_{\ell+2}$.

Figure 2 illustrates colored hypercubes for different values of d and ℓ.

At each iteration of the algorithm, we color $H_{\ell+1+i}$ by introducing 2^ℓ new colors. These new colors are used to color the joining edges. So, the colors used to color the edges of the two

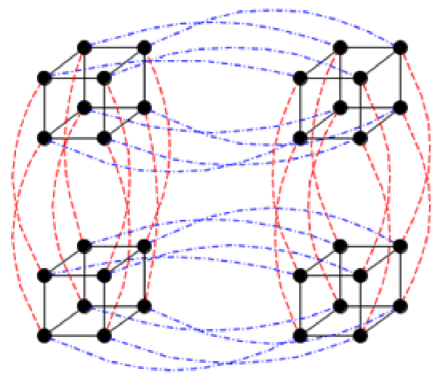

Figure 5: With $\ell = 2$, $H_{\ell+1}$ is an H_3. Hypercube H_5 is viewed as containing $2^{d-\ell-1} = 4$ $H_{\ell+1}$. At each step the joining edges join the hypercubes $H_{\ell+1}$ to their corresponding copies.

$(\ell + i)$-cubes and the joining edges are different and the coloring is an ℓ-distance edge coloring. Furthermore, we use $|E(H_{\ell+1})|$ colors to initiate the coloring of the first $H_{\ell+1}$. So, the total number of colors used to color H_d is the sum of the number of colors of the joining edges (i.e., $(d - (\ell + 1))\, 2^{\ell}$) and the number of colors of $H_{\ell+1}$ (i.e., $(\ell + 1)2^{\ell}$). Thus, $\chi'_{\ell}(H_d) \leq d2^{\ell}$.

Consequently, $\chi'_{\ell}(H_d) = d2^{\ell}$.

□

3.2 Meshes

A mesh is denoted by $M_{n,m} = G(V(G), E(G))$, where the vertex set is $V(G) = \{x_{i,j}$ with $0 \leq i \leq n-1,\ 0 \leq j \leq m-1\}$ and the edge set is $E(G) = \{(x_{i,j}, x_{i,j+1}),\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-2\} \cup \{(x_{i,j}, x_{i+1,j}), 0 \leq i \leq n-2, 0 \leq j \leq m-1\}$. We denote by $\mathcal{H}_k$, with $0 \leq k \leq n-1$, the sequence of horizontal edges of $M_{n,m}$: $\mathcal{H}_k = ((x_{k,0}, x_{k,1}), ..., (x_{k,m-2}, x_{k,m-1}))$ (cf. Figure 6). We denote by $c(\mathcal{H}_k) = (1, 2, \ldots, m-1)$ the coloring of the edges of $\mathcal{H}_k$, i.e., the first edge of $\mathcal{H}_k$ has the color 1, the second edge of $\mathcal{H}_k$ has the color 2 and so on. Similarly, we denote by $\mathcal{V}_k$, with $0 \leq k \leq n-2$, the sequence of vertical edges of the mesh $M_{n,m}$: $\mathcal{V}_k = ((x_{k,0}, x_{k+1,0}), ..., (x_{k,m-1}, x_{k+1,m-1}))$. For a coloring c of $M_{n,m}$, the notation $c(\mathcal{V}_k) = (1, 2, \ldots, m)$ means that the first edge of $\mathcal{V}_k$ has the color 1, the second edge of $\mathcal{V}_k$ has the color 2 and so on.

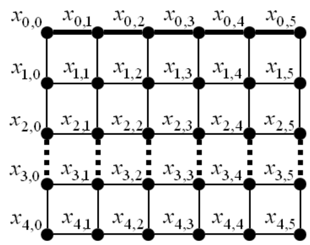

Figure 6: $M_{5,6}$ is a mesh. Bold solid edges give an example of a sequence of horizontal edges $\mathcal{H}_0 = ((x_{0,0}, x_{0,1}), (x_{0,1}, x_{0,2}), (x_{0,2}, x_{0,3}), (x_{0,3}, x_{0,4}), (x_{0,4}, x_{0,5}))$ and bold dotted edges give an example of a sequence of vertical edges $\mathcal{V}_2 = ((x_{2,0}, x_{3,0}), (x_{2,1}, x_{3,1}), (x_{2,2}, x_{3,2}), (x_{2,3}, x_{3,3}), (x_{2,4}, x_{3,4}), (x_{2,5}, x_{3,5}))$.

We first build an ℓ-ball $S_\ell \subseteq M_{n,m}$. The middle $\mathcal{C}_\ell$ of S_ℓ is a vertex satisfying the conditions stated by the following definition.

Definition 3. *Let $\ell \geq 0$, $n, m \geq \ell + 3$ and let $M_{n,m}$ be a two-dimensional mesh. Let $v = x_{i,j} \in V(M_{n,m})$ be a vertex satisfying $\lfloor \frac{\ell}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{\ell}{2} \rfloor - 1$ and $\lfloor \frac{\ell}{2} \rfloor + 1 \leq j \leq m - \lfloor \frac{\ell}{2} \rfloor - 2$ and let $v' = x_{i,j+1}$. A middle $\mathcal{C}_\ell$ of an ℓ-ball of $M_{n,m}$ is defined by:*

$$\mathcal{C}_\ell = \begin{cases} \{v\} & \text{if } \ell \textit{ is even,} \\ \{v, v'\} & \text{if } \ell \textit{ is odd.} \end{cases}$$

Note that $\mathcal{C}_\ell$ is not unique since several vertices may satisfy the condition stated by this definition. Furthermore, if ℓ is odd then to each specific middle corresponds an edge e_ℓ defined as follows:

Definition 4. *Let $n, m \geq \ell + 3$ and let $M_{n,m}$ be a two-dimensional mesh. Let $\ell \geq 0$ be odd. Let $\mathcal{C}_\ell = \{x_{i,j}, x_{i,j+1}\}$ be a middle of an ℓ-ball of $M_{n,m}$. We define the edge e_ℓ as follows:*

$$e_\ell = (x_{i+\lfloor \frac{\ell}{2} \rfloor+1,j}, x_{i+\lfloor \frac{\ell}{2} \rfloor+1,j+1}).$$

An ℓ-ball is then constructed by the following definition.

Definition 5. *Let $\ell \geq 0$, $n, m \geq \ell + 3$ and let $M_{n,m}$ be a two-dimensional mesh. Let $Q = \{u \in V(M_{n,m}) : dist(u, \mathcal{C}_\ell) \leq \lfloor \frac{\ell}{2} \rfloor + 1\}$ be the set of vertices at distance less than or equal to $\lfloor \frac{\ell}{2} \rfloor + 1$*

from a middle $\mathcal{C}_\ell$. An ℓ-ball S_ℓ of $M_{n,m}$ of middle $\mathcal{C}_\ell$ is defined by:

$$S_\ell = \begin{cases} M_{n,m}[Q] & \text{if } \ell \textit{ is even,} \\ M_{n,m}[Q] \cup \{e_\ell\} & \text{if } \ell \textit{ is odd.} \end{cases}$$

Figure 7 (respectively Figure 8) gives some examples of the construction of S_ℓ when ℓ is even (respectively ℓ is odd).

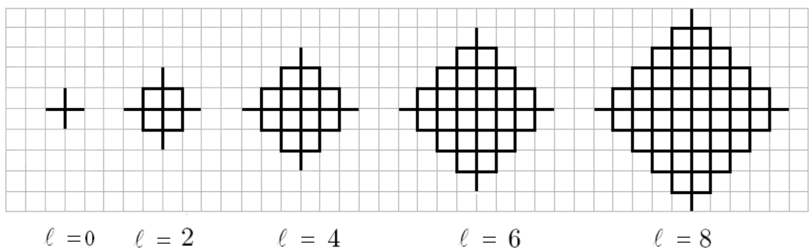

Figure 7: Examples of ℓ-balls when ℓ is even.

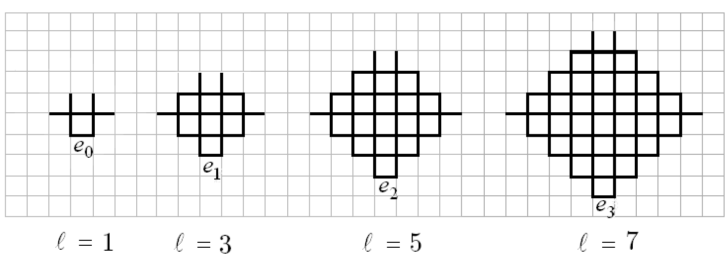

Figure 8: Examples of ℓ-balls when ℓ is odd.

The following lemma gives the size of the ℓ-ball S_ℓ:

Lemma 2. *For $n, m \geq \ell + 3$, let $S_\ell \subseteq M_{n,m}$ be an ℓ-ball of $M_{n,m}$. Then $|E(S_\ell)| = (\ell+2)^2 - (\ell \bmod 2)$.*

Proof. Suppose ℓ is even. Let $\ell = 2\ell'$. Figure 9 shows how to obtain $S_{\ell+2}$ from S_ℓ. Let us denote by $E_{\ell'}$ the number of edges in S_ℓ. Using Figure 9, we can easily verify that $E_{\ell'+1} = E_{\ell'} + 4(2(\ell'+1)+1)$ for all integers $\ell' \geq 0$. Consequently we can define an arithmetic sequence

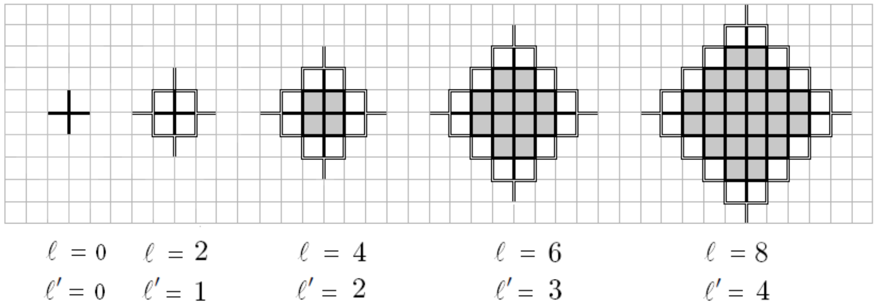

Figure 9: ℓ-ball $S_{\ell+2}$ vs S_ℓ when ℓ is even.

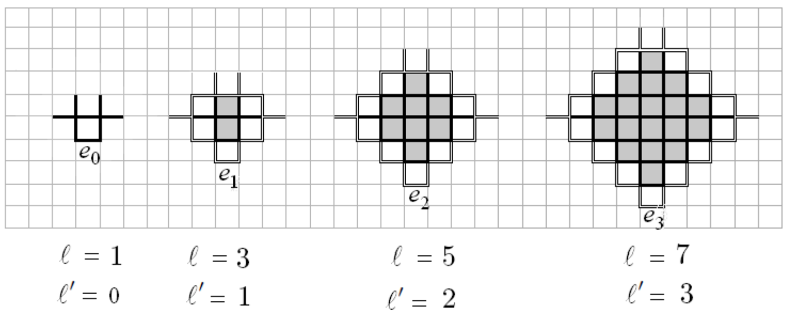

Figure 10: ℓ-ball $S_{\ell+2}$ vs S_ℓ when ℓ is odd.

and $E_{\ell'} = \sum_{i=0}^{\ell'} 4(2i+1) = (2\ell'+2)^2 = (\ell+2)^2$.

Suppose ℓ is odd. Let $\ell = 2\ell'+1$. Similar to the case when ℓ is even, we have $E_{\ell'+1} = E_{\ell'} + 8\,(\ell'+2)$ (see Figure 10). Consequently we can define an arithmetic sequence and $E_{\ell'} = \sum_{i=0}^{\ell'} 8(i+1) = (2\ell'+2)(2\ell'+4) = (\ell+2)^2 - 1$. $\square$

Theorem 2. *Let $M_{n,m}$ be a 2-dimensional mesh. Then the ℓ-chromatic index of $M_{n,m}$ is given by: $\chi'_\ell(M_{n,m}) = (\ell+2)^2 - (\ell \mod 2)$ for any $n, m \geq \ell+3$.*

Proof. To obtain the desired lower bound on $\chi'_\ell(M_{n,m})$, we consider the ℓ-ball $S_\ell \subseteq M_{n,m}$. According to Lemma 2, $|E(S_\ell)| = (\ell+2)^2 - (\ell \mod 2)$ and for any $e, e' \in E(S_\ell)$, $dist(e,e') \leq \ell$. Hence, all edges of S_ℓ must be assigned distinct colors. Consequently $\chi'_\ell(M_{n,m}) \geq (\ell+2)^2 - (\ell \mod 2)$.

For the upper bound, if ℓ is even, we define the coloring c of $\mathcal{H}_k, 0 \leq k \leq n-1$, as follows:

- If $0 \leq k \leq \frac{\ell}{2}$, we define the set of colors $Q^k = \{q_0^k, q_1^k, \ldots, q_\ell^k\}$ such that $q_i^k = k(\ell+2) + 1 + i$, with $0 \leq i \leq \ell$. We then color cyclically the edges of $\mathcal{H}_k$ with colors of Q^k (i.e., $c((x_{k,t}, x_{k,t+1})) = q^k_{t\, mod\, (\ell+1)}$, $0 \leq t \leq m-2$).

- If $\frac{\ell}{2} + 1 \leq k \leq \ell + 1$, we define the set of colors $Q'^k = \{q_0'^k, q_1'^k, ..., q_\ell'^k\}$ such that $q_i'^k = q^j_{(i+\frac{\ell}{2})\, mod\, (\ell+1)}$, with $0 \leq i \leq \ell$ and $j = k - \frac{\ell}{2} - 1$. We color cyclically the edges of $\mathcal{H}_k$ with colors of Q'^k (i.e., $c((x_{k,t}, x_{k,t+1})) = q'^k_{t\, mod\, (\ell+1)}$, $0 \leq t \leq m-2$).

- For $k > \ell + 1$, we color the edges of $\mathcal{H}_k$ by $c(\mathcal{H}_k) = c(\mathcal{H}_{k\, mod\, (\ell+2)})$.

To verify that there is no color conflict between horizontal edges, we must prove that for any $e_i = (x_{k,i}, x_{k,i+1}) \in \mathcal{H}_k$, and any $e_j = (x_{k',j}, x_{k',j+1}) \in \mathcal{H}_{k'}$, if $c(e_i) = c(e_j)$ then $dist(e_i, e_j) = |j - i - 1| + |k' - k| > \ell$. As $c(\mathcal{H}_k) = c(\mathcal{H}_{k \mod (\ell+2)})$ for $k > \ell + 1$, we focus on $k \leq \ell + 1$. For this, we consider two cases depending on k and k'.

Case $k = k'$. Two horizontal edges that have the same color are at distance $\ell + 2$ by construction.

Case $k \neq k'$. We only consider the subcase $0 \leq k \leq \frac{\ell}{2}$ and $\frac{\ell}{2} + 1 \leq k' \leq \ell + 1$ because $c(e_i) \neq c(e_j)$ for any $e_i \in \mathcal{H}_k$ and $e_j \in \mathcal{H}_{k'}$ otherwise.

According to the coloring of horizontal edges, $\mathcal{H}_k$ and $\mathcal{H}_{k'}$ have the same set of colors if they are at distance $\frac{\ell}{2}+1$. So, $k' = k+\frac{\ell}{2}+1$. For any $e_i \in \mathcal{H}_k$, $c(e_i) = 1+i+k(\ell+2)$, where $0 \leq i \leq \ell+1$. Without loss of generality, we consider the interval $\left[\frac{\ell}{2} + 1, \frac{3\ell}{2} + 3\right]$ to have all the sequence of colors Q'^k. In this case, for any $e_j \in \mathcal{H}_{k'}$, $c(e_j) = (k' - \frac{\ell}{2} - 1)(\ell+2) + j - \frac{\ell}{2} - 1$, where $\frac{\ell}{2}+1 \leq j \leq \frac{3\ell}{2}+3$. Two edges e_i and e_j have the same color means that $k(\ell+2)+i = (k' - \frac{\ell}{2} - 1)(\ell+2) + j - \frac{\ell}{2} - 1$. This gives $i = j - \frac{\ell}{2} - 1$. So, $dist(e_i, e_j) = \ell + 1 > \ell$.

If ℓ is odd, we define the coloring c of $\mathcal{V}_k, 0 \leq k \leq n-2$ as follows:

- If $0 \leq k \leq \lceil \frac{\ell}{2} \rceil$, we define the set of colors $Q^k = \{q_0^k, q_1^k, \ldots, q_\ell^k\}$ such that $q_i^k = k(\ell + 1) + 1 + i$, with $0 \leq i \leq \ell$. We then color cyclically the edges of $\mathcal{V}_k$ with colors of Q^k (i.e., $c((x_{k,t}, x_{k+1,t})) = q^k_{t\, mod\, (\ell+1)}$, $0 \leq t \leq m-1$).

- If $\lceil \frac{\ell}{2} \rceil + 1 \le k \le \ell + 2$, we define the set of colors $Q'^k = \{q'^k_0, q'^k_1, ..., q'^k_\ell\}$ such that $q'^k_i = q^j_{\left(i+\lceil \frac{\ell}{2} \rceil\right) \, mod \, (\ell+1)}$, with $0 \le i \le \ell$ and $j = k - \lceil \frac{\ell}{2} \rceil - 1$. We color cyclically the edges of $\mathcal{V}_k$ with colors of Q'^k (i.e., $c((x_{k,t}, x_{k+1,t})) = q'^k_{t \, mod \, (\ell+1)}$, $0 \le t \le m-1$).

- For $k > \ell + 2$, we color the edges of $\mathcal{V}_k$ by $c(\mathcal{V}_k) = c(\mathcal{V}_{k \, mod \, (\ell+3)})$.

Similar to the case where ℓ is even we must prove that for any $e_i = (x_{i,k}, x_{i+1,k}) \in \mathcal{V}_k$, and $e_j = (x_{j,k'}, x_{j+1,k'}) \in \mathcal{V}_{k'}$ such that $c(e_i) = c(e_j)$ then $dist(e_i, e_j) = |j - i - 1| + |k' - k| > \ell$. If $k = k'$ then $c(e_i) = c(e_j)$ implies $dist(e_i, e_j) = \ell + 1$ by construction.

If $k \neq k'$, let us consider only the subcase $0 \le k \le \lceil \frac{\ell}{2} \rceil$ and $\lceil \frac{\ell}{2} \rceil + 1 \le k' \le \ell + 2$ because $c(e_i) \neq c(e_j)$ for any $e_i \in \mathcal{V}_k$ and $e_j \in \mathcal{V}_{k'}$ otherwise. According to the coloring of vertical edges, $\mathcal{V}_k$ and $\mathcal{V}_{k'}$ have the same set of colors if they are at distance $\lceil \frac{\ell}{2} \rceil + 1$. So, $k' = k + \lceil \frac{\ell}{2} \rceil + 1$. For any $e_i \in \mathcal{V}_k$, $c(e_i) = 1 + k(\ell+1) + i$, where $0 \le i \le \ell + 2$. Without loss of generality, we consider the interval $\left[\lceil \frac{\ell}{2} \rceil + 1, \lceil \frac{3\ell}{2} \rceil\right]$ to have all the sequence of colors Q'^k used to color vertical edges. In this case, for any $e_j \in \mathcal{V}_{k'}$, $c(e_j) = 1 + (k' - \lceil \frac{\ell}{2} \rceil - 1)(\ell+1) + j - \lceil \frac{\ell}{2} \rceil - 1$, where $\lceil \frac{\ell}{2} \rceil + 1 \le j \le \lceil \frac{3\ell}{2} \rceil$. Edges e_i and e_j have the same color means that $k(\ell+1) + i = (k' - \lceil \frac{\ell}{2} \rceil - 1)(\ell+1) + j - \lceil \frac{\ell}{2} \rceil$. This gives $i = j - \lceil \frac{\ell}{2} \rceil$. So, $dist(e_i, e_j) = \ell + 1 > \ell$.

In both cases, we color the second set of edges (vertical edges for ℓ even and horizontal edges for ℓ odd) by considering the grid $M_{m,n}$ and using the same coloring with colors shifted by the value of the maximum color used for the first set of edges (i.e., $(\frac{\ell}{2} + 1)(\ell + 2)$ for ℓ even and $\left(\lceil \frac{\ell}{2} \rceil + 1\right)(\ell + 1)$ for ℓ odd). So, we use disjoint sets of colors between vertical and horizontal edges in both cases. Figures 11 and 12 show examples of this coloring for ℓ even and ℓ odd respectively.

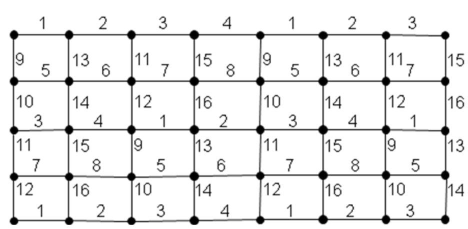

Figure 11: 2-distance edge coloring of $M_{5,8}$.

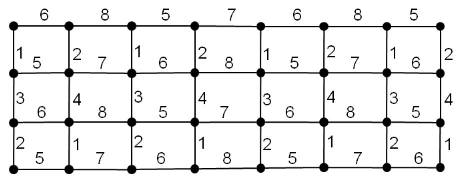

Figure 12: 1-distance edge coloring of $M_{4,8}$.

Note that to achieve this coloring we use $2\left(\frac{\ell}{2}+1\right)(\ell+2) = (\ell+2)^2$ colors in the case of ℓ even and $2\left(\left\lceil\frac{\ell}{2}\right\rceil+1\right)(\ell+1) = (\ell+2)^2-1$ colors in the case of ℓ odd.

□

4 ℓ-distance edge coloring of power graphs of trees and cycles

In this section, we focus on the ℓ-chromatic index of some classes of p^{th} power graphs. For a natural number p, the p^{th} power graph G^p is the graph obtained from graph G by adding an edge between every pair of vertices for which there exists a path of length at most p in G. The problem of coloring this class of graphs has attracted considerable attention. For example, interferences in wireless communication may be modeled by a power graph where vertices are the receivers and in which two vertices that interfere are joined by an edge. Indeed, a vertex interferes with another vertex at distance ℓ if and only if it interferes with all vertices between them. We present here bounds and exact values for the ℓ-chromatic index of power graphs of trees and cycles.

4.1 Power graphs of trees

A k-ary tree is a rooted tree with at most k children for each internal vertex. A complete k-ary tree is a k-ary tree where all *leaves* are at the same depth and internal vertices have exactly k children. We first consider the ℓ-chromatic index of power graphs of paths (complete k-ary trees for $k = 1$) and we investigate power graphs of complete k-ary trees with $k \geq 2$. Then, we deduce an upper bound for the power graph of a general tree.

4.1.1 Power graph of paths

Let P_n^p be the p^{th} power of a path P_n of order n. We start by evaluating the size of $S_\ell(P_n^p)$.

Lemma 3. *Let P_n be a path of order $n \geq p(\ell+2)+1$. For any $p \geq 1$ and $\ell \geq 0$, we have*

$$|E(S_\ell(P_n^p))| = \begin{cases} 2p & \text{if } \ell = 0\ , \\ p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p & \text{otherwise.} \end{cases}$$

Proof. The set of vertices of P_n is given by $V(P_n) = \{x_1, x_2, \ldots, x_n\}$. According to its definition, an ℓ-ball of P_n^p is the subgraph S_ℓ induced by the edges incident to a sequence Q of $p\ell+1$ vertices such that $Q = \{x_{t+1}, x_{t+2}, \ldots, x_{t+p\ell+1}\}$ with $p \leq t \leq (n - p(\ell+1) - 1)$. Thus, if $\ell = 0$, then $|Q| = 1$. The number of incident edges is thus equal to $2p$. Moreover, if $\ell > 0$ then $|Q| = p\ell+1$. The degree of each vertex of Q in S_ℓ is equal to $\Delta(P_n^p) = 2p$ and for the vertices of $V(S_\ell)\backslash Q$, we have two vertices with degree 1, two vertices with degree $2, \ldots,$ two vertices with degree p (cf. Figure 13). The number of edges in S_ℓ is thus equal to $|E(S_\ell)| = \frac{1}{2}\sum_{i=1}^{p(\ell+2)+1} d(x_i) = \frac{1}{2}\left(2\sum_{j=1}^{j=p} j + 2p(p\ell+1)\right) = p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p$. □

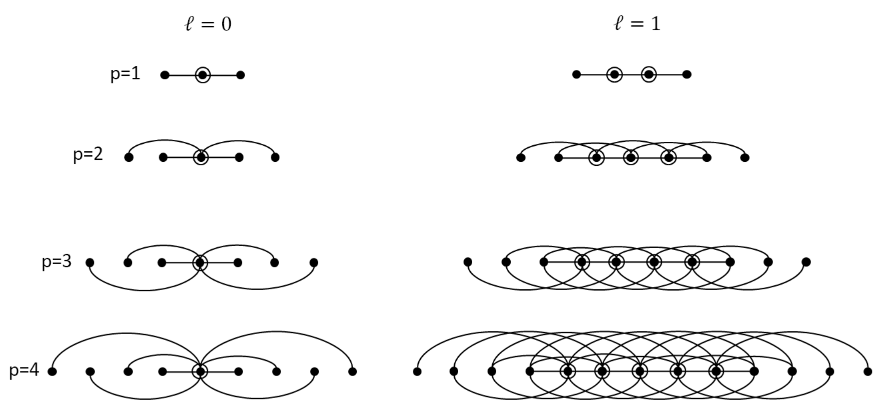

Figure 13: Examples of some ℓ-balls for a power graph of a path where $0 \leq \ell \leq 1$ and $1 \leq p \leq 4$. Circled vertices are vertices of the set Q.

Then, we deduce the ℓ-chromatic index of P_n^p.

Theorem 3. *Let P_n be the path on vertices $x_1, x_2, \ldots, x_n$ with $n \geq p(\ell+2)+1$. For any $p \geq 1$ and $\ell \geq 0$, the ℓ-chromatic index of the power graph P_n^p is given by*

$$\chi_\ell'(P_n^p) = |E(S_\ell)| = \begin{cases} 2p & \text{if } \ell = 0\ , \\ p^2\left(\ell + \frac{1}{2}\right) + \frac{3}{2}p & \text{otherwise.} \end{cases}$$

Proof. The lower bound $\chi_\ell'(P_n^p) \geq |E(S_\ell)|$ is obvious since two different edges of S_ℓ are at distance at most ℓ. We prove the upper bound by construction. We decompose P_n^p into p sets of edges (denoted E_i) where each set corresponds to a power (see examples in Figure 14). Thus, E_i is the set of edges of P_n^p such that $E_i = \{(x_1, x_{i+1}), (x_2, x_{i+2}), \ldots, (x_{n-i}, x_n)\}$. Then, if $\ell = 0$, we color the edges of the power graph with $\Delta(P_n^p)$ colors. Indeed, the graph $G_i = (V(P_n), E_i)$ is a bipartite graph [23] where $\Delta(G_i) = 2$. We thus color each E_i, with $i \in \{1, 2, \ldots, p\}$, with two different colors. So, if $\ell = 0$ then the ℓ-chromatic index is equal to $\chi_0'(P_n^p) = 2p$. If $\ell > 0$, we define the coloring of each set of edges as follows: we color cyclically the edges of E_i with $1 + i + p\ell$ colors. We note that the colors used for E_i are different from the colors of E_j, with $i \neq j$. Moreover, for a given E_i, the distance between edges that share the same color is equal to $\left\lceil \frac{1+i+p\ell}{p} \right\rceil > \ell$. Thus, the coloring is an ℓ-distance edge coloring. The number of colors used is then given by $\sum_{i=1}^{p}(1 + i + p\ell) = p^2\left(\ell + \frac{1}{2}\right) + \frac{3}{2}p$. Therefore, $\chi_\ell'(P_n^p) = p^2\left(\ell + \frac{1}{2}\right) + \frac{3}{2}p$. $\square$

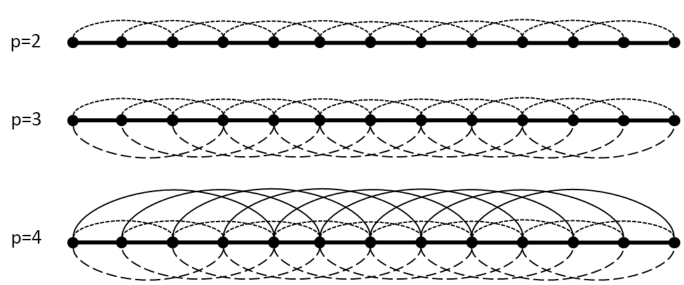

Figure 14: Some power graphs of the path P_{13} (marked by bold edges). Dotted edges, dashed edges and solid edges denote respectively edges of power 2, power 3 and power 4 (i.e., E_2, E_3 and E_4).

4.1.2 Power graph of k-ary trees, with $k \geq 2$

Let h be the height of a k-ary tree T. Each vertex of T is denoted $x_{i,j}$, where the index i, $0 \leq i \leq h$, indicates the level of the vertex and j, $0 \leq j \leq k^i - 1$, is the position of the vertex within a given level i (the root of the tree is situated at level 0 and is noted $x_{0,0}$). For the p^{th} power of T, denoted T^p, we use the same notations (height, level and position), since the complete k-ary tree T is included in T^p. We note that for all the following figures, bold edges are the edges of the complete k-ary tree T and thin edges are the power edges of T^p.

We first remark that for low height trees, the ℓ-chromatic index is easily deduced.

Theorem 4. *Let T be a complete k-ary tree of height $h < \left\lfloor \frac{p\ell}{2} \right\rfloor + 1$. Then the ℓ-chromatic index of T^p is $\chi'_\ell(T^p) = |E(T^p)|$, with $p \geq 1$, $k \geq 2$ and $\ell \geq 0$.*

Proof. If $h < \left\lfloor \frac{p\ell}{2} \right\rfloor + 1$, the distance between any two edges in the p^{th} power graph of T is lower than $\ell + 1$. Consequently, no color can be repeated and $\chi'_\ell(T^p) = |E(T^p)|$. □

For large height trees, i.e., $h \geq p(\ell + 2)$, we first compute the size of $S_\ell(T^p)$. First note that we distinguish two cases for the topology of $S_\ell(T^p)$ according to the parity of $p\ell$. Indeed, we can see that the vertices of S_ℓ are organized around a center if $p\ell$ is even and around a bicenter if $p\ell$ is odd. Figure 15 shows examples of S_ℓ in both cases. Furthermore, $S_\ell(T^p)$ is composed of two distinct kind of vertices (see Figure 15):

- *Internal vertices*: vertices with the same degree D as the middle of S_ℓ where $D = (1 + k)\sum_{i=0}^{i=p-1} k^i = \frac{1+k}{1-k}(1 - k^p)$. The distance between these vertices and the middle is at most $\left\lfloor \frac{p\ell}{2} \right\rfloor$.
- *Outlying vertices*: an outlying vertex is at distance $\left(\left\lfloor \frac{p\ell}{2} \right\rfloor + 1 + j\right)$ from the middle of S_ℓ. It has degree $\widetilde{D_j} = 1 + \sum_{i=0}^{i=p-j-2} k^j = 1 + \frac{1-k^{p-j-1}}{1-k}$.

Then, we have the following result:

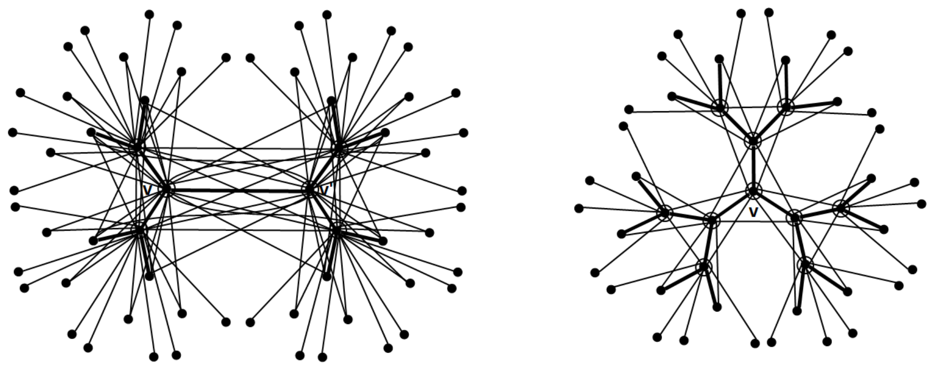

Figure 15: Two examples of ℓ-balls if $h \geq p(\ell+2)$. **(a)** $k=2$, $\ell=1$, $p=3$ and $\{v, v'\}$ is the middle of S_ℓ, **(b)** $k=2$, $\ell=2$, $p=2$ and v is the middle of S_ℓ. Circled vertices are vertices with the same degree as the middle (i.e., internal vertices).

Theorem 5. *Let T be a complete k-ary tree of height $h \geq p(\ell+2)$. For any $p \geq 1$ and $\ell \geq 0$, the size of $S_\ell(T^p)$ is*

$$|E(S_\ell(T^p))| = \begin{cases} \frac{D}{1-k}\left(1-k^{\lfloor \frac{p\ell}{2} \rfloor+1}\right) + \sum_{i=0}^{i=p-1} \widetilde{D}_i k^{\lfloor \frac{p\ell}{2} \rfloor+i+1} & \text{if } p\ell \text{ is odd,} \quad (a) \\ \frac{D}{2}\left(1+\frac{1+k}{1-k}\left(1-k^{\frac{p\ell}{2}}\right)\right) + \frac{1+k}{2}\sum_{i=0}^{i=p-1} \widetilde{D}_i k^{\frac{p\ell}{2}+i} & \text{otherwise.} \quad (b) \end{cases}$$

Proof. We present each case separately.

Case (a) The number of internal vertices of S_ℓ is $N_D = 2\sum_{i=0}^{i=\lfloor \frac{p\ell}{2} \rfloor} k^i = \frac{2}{1-k}\left(1-k^{\lfloor \frac{p\ell}{2} \rfloor+1}\right)$. The degree of outlying vertices depends on their distance to the bicenter of $S_\ell(T^p)$. There are $N_{\widetilde{D}_i} = 2k^{\lfloor \frac{p\ell}{2} \rfloor+i+1}$ vertices at distance $\left\lfloor \frac{p\ell}{2} \right\rfloor + i + 1$ to the bicenter. Each one has degree $\widetilde{D}_i$, where $0 \leq i \leq p-1$. Since $\sum_{x_i \in V(G)} d(x_i) = 2|E(G)|$, then $|E(S_\ell)| = \frac{1}{2}\left(DN_{\widetilde{D}_i} + \sum_{i=0}^{i=p-1} \widetilde{D}_i N_{\widetilde{D}_i}\right) = \frac{D}{1-k}\left(1-k^{\lfloor \frac{p\ell}{2} \rfloor+1}\right) + \sum_{i=0}^{i=p-1} \widetilde{D}_i k^{\lfloor \frac{p\ell}{2} \rfloor+i+1}$.

Case (b): We use the same reasoning as in case (a). The number of internal vertices is $N_D = 1 + (1+k)\sum_{i=0}^{i=\frac{p\ell}{2}-1} k^i = 1 + \frac{1+k}{1-k}\left(1-k^{\frac{p\ell}{2}}\right)$ and there are $N_{\widetilde{D}_i} = (k+1)\,k^{\frac{p\ell}{2}+i}$ outlying vertices at distance $\frac{p\ell}{2} + i + 1$ to the center. Each one has degree $\widetilde{D}_i$, where $0 \leq i \leq p-1$. Then, $|E(S_\ell)| = \frac{1}{2}\left(DN_{\widetilde{D}_i} + \sum_{i=0}^{i=p-1} \widetilde{D}_i N_{\widetilde{D}_i}\right) = \frac{D}{2}\left(1+\frac{1+k}{1-k}\left(1-k^{\frac{p\ell}{2}}\right)\right) + \frac{1+k}{2}\sum_{i=0}^{i=p-1} \widetilde{D}_i k^{\frac{p\ell}{2}+i}$. $\square$

Thus, we deduce the ℓ-chromatic index of T^p.

Theorem 6. *Let T be a complete k-ary tree of height $h \geq p\,(\ell+2)$ and let $S_\ell(T^p)$ be an ℓ-ball of T^p). If $p \geq 1$ and $\ell \geq 1$, then $\chi'_\ell(T^p) = |E(S_\ell(T^p))|$.*

Proof. The definition of $S_\ell(T^p)$ implies obviously that $\chi'_\ell(T^p) \geq |E(S_\ell(T^p))|$. We prove the upper bound by construction. Consider the case of $p\ell$ even where $S_\ell(T^p)$ has a center (the same reasoning can be done for the case of $p\ell$ odd). We first note that $S_\ell(T^p)$ is not unique since every vertex $x_{i,j}$, with $\left\lfloor \frac{p\ell}{2} \right\rfloor + p \leq i \leq h - \left\lfloor \frac{p\ell}{2} \right\rfloor - p$ and $0 \leq j \leq k^i - 1$, can be a center of $S_\ell(T^p)$. Hence, $S_\ell(T^p) \equiv S_\ell(x_{a,b})$ such that $|S_\ell(x_{a,b})| = \max\{|S_\ell(x_{i,j})|, 0 \leq i \leq h \text{ and } 0 \leq j \leq k^i - 1\}$. The coloring of $S_\ell(T^p)$ uses a set of $|E(S_\ell)|$ colors $C = \{1, 2, \ldots, |E(S_\ell)|\}$. To color T^p, we traverse the graph in level-order and for each visited vertex $x_{i,j}$, we color $S_\ell(x_{i,j})$ with different colors. As $|S_\ell(x_{i,j})| \leq |S_\ell(T^p)|$, there exist sufficient colors in C to properly color $S_\ell(x_{i,j})$.

We can see that two subgraphs $S_\ell(x_{i,j})$ and $S_\ell(x_{i',j'})$ can have neighbor edges (between their middles and lower levels) (see Figure 16 where $S_1(x_{3,3})$ and $S_1(x_{3,1})$ have neighbor edges $(x_{3,3}, x_{1,0})$ and $(x_{3,1}, x_{1,0})$). If these subgraphs are colored separately, these edges can have the same color. However, by coloring the subgraphs $S_\ell(x_{i,j})$ according to a breadth first traversal of T^p, the edges between $x_{i,j}$ and vertices in lower levels are already properly colored. Thus, the coloring in higher levels is a proper coloring and since no color is used twice in any subgraph S_ℓ the ℓ-distance constraint is satisfied. Therefore $\chi'_\ell(T^p) \geq |E(S_\ell(T^p))|$. $\square$

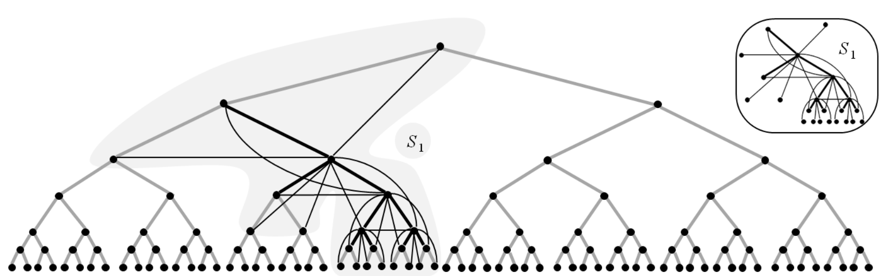

Figure 16: Example of ℓ-ball $S_\ell \subset T^p$ where $k = 2$, $\ell = 1$, $p = 2$ and $h = 6$. The center of S_ℓ is the vertex $v = x_{3,3}$. Dark edges, both bold and thin, are the edges of S_ℓ, the thin ones are power edges. For readability of the figure, we draw only the power edges that belong to S_ℓ.

For the remaining cases, i.e., $\left\lfloor \frac{p\ell}{2} \right\rfloor + 1 \leq h < p(\ell+2)$, the size of S_ℓ is smaller than the size of the possible maximum subgraph described above. The above result gives an upper bound for these remaining cases. Moreover, this result proposes also an upper bound for general trees.

Corollary 1. *Let T a tree of height h. For $p \geq 1$ and $\ell \geq 0$, we have $\chi_\ell^{'}(T^p) \leq \chi_\ell^{'}(T^p_{\Delta(T^p)-1})$, where $T_{\Delta(T^p)-1}$ is a complete $(\Delta(T^p)-1)$-ary tree of height h.*

Proof. This corollary is deduced from Theorem 5 since T^p is an induced subgraph of $T^p_{\Delta(T^p)-1}$. □

4.2 Power graph of cycles

Let C_n^p be the p^{th} power graph of a cycle C_n, with $p \geq 1$. We first note that for $n \leq 2p+1$, we have $C_n^p = C_n^{p+1} = C_n^{p+2} = \cdots = K_n$ where K_n is the complete graph with n vertices and therefore we have:

$$\chi'_\ell(C_n^p) = \chi'_\ell(C_n^{p+1}) = \chi'_\ell(C_n^{p+2}) = \cdots = \chi'_\ell(K_n) = \frac{n(n-1)}{2}.$$

When $n > 2p+1$, we have the following result for $\ell = 0$:

Proposition 1. *Let $p \geq 1$, $\ell = 0$ and let C_n be a cycle of order $n > 2p+1$. Then*

$$\chi'_0(C_n^p) = 2p+1 \text{ if } n \text{ is odd,}$$

$$2p \leq \chi'_0(C_n^p) \leq 2p+1 \text{ otherwise.}$$

Proof. If n is odd, since C_n^p is a regular graph with an odd number of vertices, the graph C_n^p is of class two [34]. Consequently, $\chi'_0(C_n^p) = 2p+1$. If n is even, then Vizing's proof [33] gives the result. □

Then, we focus on the case $\ell > 0$. First, we consider small cycles.

Theorem 7. *Let C_n be a cycle of order n, with $3 \leq n \leq 2p\ell+3$. Then $\chi_\ell^{'}(C_n^p) = |E(C_n^p)|$, where $p \geq 1$ and $\ell > 0$.*

Proof. According to the definition of an ℓ-distance edge coloring, the distance between two edges with the same color is at least $\ell+1$. So, if $n \leq 2\ell p+3$ the distance between any two edges in the p^{th} power graph of C_n is smaller than $\ell+1$. Consequently, no color can be repeated and $\chi'_\ell(C_n^p) = |E(C_n^p)|$. □

In the following results, we give some values for the ℓ-chromatic index of the power graph C_n^p where $n > 2\ell p+3$. We first present an upper bound on the ℓ-chromatic index.

Theorem 8. *Let C_n be a cycle of order $n > 2p\ell+3$. Then for any $p \geq 1$ and $\ell > 1$, we have $\chi'_\ell(C_n^p) \leq p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p+\sum_{i=1}^{p}\left\lceil \frac{r_i}{q_i}\right\rceil$, where $q_i = \left\lfloor \frac{n}{1+i+p\ell}\right\rfloor$ and $r_i = n - q_i\left(1+i+p\ell\right)$ for every $1 \leq i \leq p$.*

Proof. We prove this upper bound by construction. The coloring of C_n^p is similar to the coloring of P_n^p in Theorem 3. We can use this coloring algorithm since $2p\ell+4 > p\left(\ell+2\right)+1$. We color cyclically, $q_i - b_i$ times (where $b_i = r_i - q_i\left\lfloor \frac{r_i}{q_i}\right\rfloor$) the edges of E_i (i.e., the set of edges of C_n^p such that $E_i = \{(x_1, x_{i+1}), (x_2, x_{i+2}), \ldots, (x_{n-i}, x_n), (x_n, x_1\})$ with $1+i+p\ell+\left\lfloor \frac{r_i}{q_i}\right\rfloor$ different colors. For the remaining edges (if $b_i \neq 0$), we color cyclically (b_i times) the edges of E_i with $1+i+p\ell+\left\lceil \frac{r_i}{q_i}\right\rceil$ colors. We note that all sets of edges have different colors. The distance between edges with the same color on the same E_i is greater than ℓ since we color cyclically with at least $1+i+p\ell+\left\lfloor \frac{r_i}{q_i}\right\rfloor$ colors. Then the number of colors is equal to $\sum_{i=1}^{p}(1+i+p\ell+\left\lceil \frac{r_i}{q_i}\right\rceil) = \chi'_\ell(P_n^p)+\sum_{i=1}^{p}\left\lceil \frac{r_i}{q_i}\right\rceil$. □

Then, we propose two particular cases where the relation $\chi'_\ell(C_n^p) = \chi'_\ell(P_n^p)$ is satisfied.

Theorem 9. *For any $p \geq 1$ and $\ell > 1$, let C_n be the cycle of order $n > 2p\ell+3$, such that*

$$n \bmod \prod_{i=1}^{i=p}(1+i+p\ell) = 0.$$

Then we have $\chi'_\ell(C_n^p) = \chi'_\ell(P_n^p)$.

Proof. Since P_n^p is a partial graph of C_n^p, then $\chi'_\ell(C_n^p) \geq \chi'{}_\ell(P_n^p)$. Then, suppose that n is a multiple of the product of the number of colors of all E_i, i.e., $n \bmod \prod_{i\in I}(1+i+p\ell) = 0$ with $I = \{1, 2, \ldots, p\}$. Then, from the definition of q_i and r_i in Theorem 8, for any $i \in I$, we have

$q_i = \left\lfloor \frac{n}{1+i+p\ell} \right\rfloor = \prod_{j\in I\setminus\{i\}}(1+j+p\ell)$ and $r_i = n - q_i\,(1+i+p\ell) = 0$. Thus, from Theorem 8 we deduce $\chi'_\ell(C_n^p) \leq p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p = \chi'_\ell(P_n^p)$. □

Theorem 10. *For any $p \geq 1$ and $\ell > 1$, let C_n be the cycle of order $n > 2p\ell+3$, such that*

$$n \bmod \left(p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p\right) = 0.$$

Then we have $\chi'_\ell(C_n^p) = \chi'_\ell(P_n^p)$.

Proof. Since P_n^p is a partial graph of C_n^p, then $\chi'_\ell(C_n^p) \geq \chi'{}_\ell(P_n^p)$. Then, suppose that $n \bmod (p^2(\ell+\frac{1}{2})+\frac{3}{2}p) = 0$. We prove the upper bound by construction. We color cyclically each set E_i with the colors of the set $\mathcal{C} = \{1,2,\ldots,p^2(\ell+\frac{1}{2})+\frac{3}{2}p\}$ as follows: color the edges of E_i cyclically with the set of colors $\{p\ell(i-1)+\frac{i}{2}(i+3)-1, p\ell(i-1)+\frac{i}{2}(i+3),\ldots,p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p, 1, 2, \ldots, p\ell(i-1)+\frac{i}{2}(i+3)-2\}$. Note that for each set E_i the edges are traversed in the same order from the vertex x_1 (i.e., we first color (x_1,x_2) in E_1, (x_1,x_3) in E_2, etc). Since $n \bmod \left(p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p\right) = 0$, the distance between edges with the same color on the same E_i is greater than ℓ. Then we prove that the distance between edges with the same color in different sets E_i and E_j is also greater than ℓ. Let $e_i = (x_a, x_{a+i}) \in E_i$ and $e_j = (x_b, x_{b+j}) \in E_j$ where $j < i \leq p$. According to the coloring algorithm, $c\,(e_i) = c\,(e_j)$ means that $a + p\ell(i-1)+\frac{i}{2}(i+3)-1 = b+p\ell(j-1)+\frac{j}{2}(j+3)-1$. Since $j < i < p$, then $b - a \geq p\ell + j + 2$ and $dist_{C_n^p}(e_i,e_j) = \left\lceil \frac{b-a-i}{p} \right\rceil \geq \ell+1$. The coloring is then an ℓ-distance edge coloring and $\chi'_\ell(C_n^p) \leq p^2\left(\ell+\frac{1}{2}\right)+\frac{3}{2}p = \chi'_\ell(P_n^p)$. □

5 Conclusion

In this paper, we considered distance edge coloring, a generalization of proper edge coloring. This coloring helps to ensure TDMA scheduling in mobile wireless sensor networks and avoid an energy consuming dynamic timeslot assignment. We studied the feasibility of this coloring in several communication topologies such as meshes, hypercubes and some power graphs. We also provided coloring algorithms for these networking structures.

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