INVARIANT GAMES

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Abstract. In the context of 2-player removal games, we define the notion of invariant game for which each allowed move is independent of the position it is played from. We present a family of invariant games which are variations of Wythoff's game. The set of P-positions of these games are given by a pair of complementary Beatty sequences related to the irrational quadratic number $\alpha_k = (1; \overline{1, k})$. We also provide a recursive characterization of this set.

We assume that the reader has some knowledge in combinatorial game theory. Basic definitions can be found in [2]. The set of nonnegative (resp. positive) integers is denoted by $\mathbb{N}$ (resp. $\mathbb{N}_{\geq 1}$).

Given an infinite sequence $S = (A_n, B_n)_{n\geq 0}$ of nonnegative integers with $(A_0, B_0) = (0, 0)$, a 2-player removal game on two heaps having S as set of P-positions can always be defined. Indeed, the following naïve rules can be chosen: from any position (x, y) not in S, there is a unique allowed move $(x, y) \to (0, 0)$. And from any position $(A_n, B_n) \in S$, any move is allowed except those leading to another position in S. Such a definition for the rules is not satisfying. In general, game rules that are considered are those which can "*easily be understood by a child*". These considerations are fuzzy. Nowadays, there is no clear formal framework to decide the quality of given game rules.

We here propose an answer to this issue by introducing the notion of *invariant games*. An invariant game has rules that are independent of the actual position of the game.

Definition 1. Consider a two-player impartial removal game G played on $\ell \geq 1$ piles of tokens. Positions and moves are thus coded by ℓ-tuples of nonnegative integers. For two ℓ-tuples $x = (x_1, \ldots, x_\ell)$ and $y = (y_1, \ldots, y_\ell)$, we write $x \prec y$ if $x_i \leq y_i$ for all $i = 1, \ldots, \ell$. The game G is *invariant*, if for all positions $p = (p_1, \ldots, p_\ell)$ and $q = (q_1, \ldots, q_\ell)$ and any move $x = (x_1, \ldots, x_\ell)$ such that $x \prec p$ and $x \prec q$ then, the move $p \to p - x$ is allowed if and only if the move $q \to q - x$ is allowed.

Otherwise stated, a game is invariant if the same moves can be played from any position, with the only restriction that enough tokens on the different piles are available.

We denote by $\mathcal{M}_G \subseteq \mathbb{N}^\ell$ the set of moves of G. If G is invariant, then the knowledge of $\mathcal{M}_G$ is enough to play the game. On the other hand, a game for which at least one move depends on the actual position is called *variant*. In a variant game, some positions are associated with specific subsets of $\mathcal{M}_G$.

Example 1. For instance, the game of Nim [3] or Wythoff's game [15] are invariant games. In Wythoff's game W, we have

$$\mathcal{M}_W = \{(i, 0) \mid i \geq 1\} \cup \{(0, j) \mid j \geq 1\} \cup \{(i, i) \mid i \geq 1\}$$

and for the game of Nim on ℓ piles,

$$\mathcal{M}_N = \{(i,0,\dots,0) \mid i \geq 1\} \cup \{(0,i,0,\dots,0) \mid i \geq 1\} \cup \cdots \cup \{(0,\dots,0,i) \mid i \geq 1\}.$$

Other invariant games are given in [5, 8, 12, 13] or also the *subtraction games* found in [2].

Example 2. Games like the *Raleigh game* [10], *the Rat and the Mouse* game [11], *Tribonacci game* [6] or *Cubic Pisot games* [7] are variant. Nevertheless, these games remain appealing, since the dependence of the game rules to the actual positions is restricted to some simple logical formula.

One can however wonder if there exist invariant games having the same sets of P-positions as those in Example 2. More generally, for any sequence $S : \mathbb{N} \to \mathbb{N}^\ell$, is there an invariant game having S as set of P-positions? The answer is negative. As an example, consider any sequence $S = (A_n, B_n)_{n\geq 0}$ starting with $(0,0),(1,2),(3,5),(4,6)$ and such that $\{A_n \mid n \geq 1\}$ and $\{B_n \mid n \geq 1\}$ make a partition of $\mathbb{N}_{\geq 1}$. There is no invariant game having S as set of P-positions because from the N-position $(1,1)$, one must play to $(0,0)$. Hence the move $(1,1)$ belongs to the set of rules. But playing from $(4,6)$ to $(3,5)$ is not allowed (there is no move between two P-positions).

We already know that Wythoff's game is invariant. Notice that its set of P-positions is given by a pair of complementary (homogeneous) Beatty sequences [1]. A pair of complementary homogeneous Beatty sequences is of the form $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)_{n\geq 1}$, with $\alpha > 1$ an irrational number, and $\beta = \alpha/(\alpha-1)$. Non-homogeneous Beatty sequences are those of the form $(\lfloor n\alpha + a \rfloor, \lfloor n\beta + b \rfloor)_{n\geq 1}$, with a and b any two real nonzero numbers.

Indeed, it is proved in [15] that the nth P-position of Wythoff's game is $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$, where τ is the golden ratio. In [9], Fraenkel investigates an invariant extension of Wythoff's game where the set of P-positions is also given by a pair of complementary Beatty sequences build over the quadratic irrational number having $(1;\overline{k})$, with $k \in \mathbb{N}_{\geq 1}$, as continued fraction expansion.

In this paper, we consider the sequence $S = (A_n, B_n)_{n\geq 0}$ build over the quadratic irrational number α_k having $(1;\overline{1,k})$ as continued fraction expansion and then show that there exists an invariant game having S as set of P-positions. This result is a step towards the following general conjecture. In the rest of the paper, assume that k is fixed once and for all.

Conjecture 1. *Given a pair of complementary Beatty sequences $S = (A_n, B_n)_{n\geq 1}$, there exists an invariant game having $S \cup \{(0,0)\}$ as set of P-positions.*

The converse does not hold. As shown in [5, 8], there are invariant games whose set of P-positions cannot be described with a pair of Beatty sequences. Notice that the invariant game discussed in [12] has a set of P-positions given by a pair of non-homogeneous Beatty sequences.

This paper is articulated as follows. In the next section, we study some particular properties of the sequence $(\lfloor n\alpha_k \rfloor)_{n\geq 0}$ using the fact that $(\lfloor (n+1)\alpha_k \rfloor - \lfloor n\alpha_k \rfloor)_{n\geq 0}$ is a Sturmian sequence, see Definition 2. In a second part, we present a family of invariant games which are variations of Wythoff's game. We obtain two characterizations of the set of P-positions. The first one is recursive (Theorem 3). The other one is based on the results given in the first section and it expresses the set of P-position using a pair of complementary Beatty sequences based on α_k (Theorem 4). From the point of view of combinatorics on words, these two theorems provide a recursive definition of a family of Sturmian words.

1. Some technical results

Let $k \geq 1$ be an integer. Let α_k be the quadratic irrational number having $(1;\overline{1,k})$ as continued fraction expansion and β_k be such that $\alpha_k^{-1} + \beta_k^{-1} = 1$. We have thus defined

$$(1)\ \ \alpha_k = 1 + \frac{\sqrt{k^2+4k}-k}{2} \in [\frac{1+\sqrt{5}}{2}, 2) \ \text{ and } \ \beta_k = \frac{3}{2} + \frac{\sqrt{k^2+4k}}{2k} \in (2, \frac{3+\sqrt{5}}{2}].$$

which are represented as functions of k in Fig. 1. The sequences $(\lfloor n\alpha_k \rfloor)_{n\geq 1}$ and

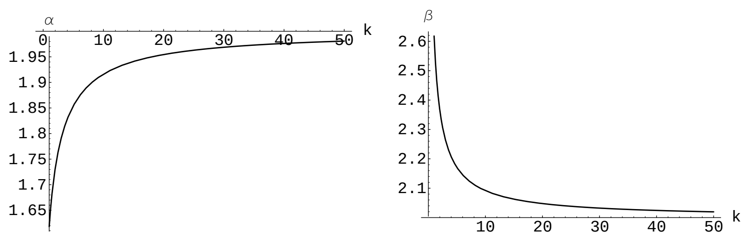

Figure 1. α_k and β_k as functions of k

$(\lfloor n\beta_k \rfloor)_{n\geq 1}$ are complementary *Beatty sequences* giving a partition of $\mathbb{N}_{\geq 1}$ [1]. Notice that for $k = 1$, α_1 is exactly the golden ratio.

Definition 2. For any positive real number γ, we write

$$\Delta_\gamma(n) := \lfloor (n+1)\gamma \rfloor - \lfloor n\gamma \rfloor.$$

It is well-known (see for instance [14]) that for any irrational number γ the first difference sequence $(\Delta_\gamma(n))_{n\geq 1}$ is a *Sturmian sequence* over $\{\lfloor\gamma\rfloor, \lfloor\gamma\rfloor + 1\}$: for all $\ell \geq 0$, there are exactly $\ell + 1$ distinct blocks of ℓ consecutive elements in the sequence, i.e., for all $\ell \geq 0$

$$\#\{\Delta_\gamma(i+1)\cdots\Delta_\gamma(i+\ell) \mid i \geq 0\} = \ell + 1.$$

The results of this section describe some properties of the sequences $(\Delta_{\alpha_k}(n))_{n\geq 1}$ and $(\Delta_{\beta_k}(n))_{n\geq 1}$. The proofs use only elementary methods but some caution is needed.

Remark 1. The sequence $(\Delta_{\alpha_k}(n))_{n\geq 1}$ is a Sturmian sequence over $\{1, 2\}$. As an example, for $k = 2$, the first elements in $(\Delta_{\alpha_2}(n))_{n\geq 1}$ are

$$22122212221221222122212221221222122212221221222122\cdots$$

For all $k \geq 1$, the three factors 21, 12 and 22 appear in the sequence $(\Delta_{\alpha_k}(n))_{n\geq 1}$ but 11 does not occur. Proceed by contradiction and assume that there exists n such that $\Delta_{\alpha_k}(n) = 1$ and $\Delta_{\alpha_k}(n+1) = 1$. Adding these two relations gives

$$\sqrt{k^2+4k} - k = \{(n+2)\,\alpha\} - \{n\,\alpha\}.$$

The l.h.s. is greater than 1 for all $k \geq 1$ (α_k is increasing with respect to k) but the r.h.s. is in $(-1, 1)$ providing the contradiction. (See Remark 2 to have the description of the $k + 1$ factors of length k.)

The same observation can be made for all $k \geq 1$, the sequence $(\Delta_{\beta_k}(n))_{n\geq 1}$ over $\{2,3\}$ for which the factor 33 does not occur. For $k = 2$, the first elements in $(\Delta_{\gamma_2}(n))_{n\geq 1}$ are

$$232232232322322322323223223223232232232322322322232\cdots$$

In what follows, we assume that k is implicitly understood as a parameter for α_k and β_k and we omit reference to k in the corresponding notation α and β.

Lemma 1. *Let α, β given in* (1) *and $n \geq 1$ be such that $\Delta_\alpha(n) = 1$. Then $\Delta_\beta(n) = 2$.*

Proof. We assume that $\alpha - \{(n+1)\,\alpha\} + \{n\,\alpha\} = 1$ and we have to show that $\beta - \{(n+1)\,\beta\} + \{n\,\beta\} = 2$. For the sake of simplicity, we set

$$\gamma := \frac{\sqrt{k^2+4k}}{2}. \tag{2}$$

Since $(k+1)^2 < k^2+4k < (k+2)^2$, it is easy to deduce that $\lfloor\gamma\rfloor = \lceil k/2\rceil$. Notice that if k is even, then for all $n \geq 1$, $\{n\,\alpha\} = \{n\,\gamma\}$. If k is odd, then for all $n \geq 0$, we have

$$\{2n\,\alpha\} = \{2n\,\gamma\} \text{ and } \{(2n+1)\,\alpha\} = \left\{\{(2n+1)\,\gamma\} + \frac{1}{2}\right\}.$$

Now let's turn our attention to expressions involving β. For all $n \geq 0$, we have

$$\{2n\,\beta\} = \{2n\,\gamma/k\} \text{ and } \{(2n+1)\,\beta\} = \left\{\{(2n+1)\,\gamma/k\} + \frac{1}{2}\right\}.$$

For all $n \geq 1$, if we consider the Euclidian division $\lfloor n\,\gamma\rfloor = q_n\,k + r_n$ with $r_n \in \{0,\ldots,k-1\}$, we get

$$\lfloor n\,\gamma/k\rfloor = q_n \text{ and } \{n\,\gamma/k\} = \frac{1}{k}\{n\,\gamma\} + \frac{r_n}{k}.$$

Notice that

$$\beta = 2 + \frac{\alpha-1}{k}. \tag{3}$$

Case 1. Assume first n odd and k even. We have

$$\begin{aligned}\beta - \{(n+1)\beta\} + \{n\beta\} &= 2 + \tfrac{\alpha-1}{k} - \{(n+1)\,\gamma/k\} + \left\{\{n\,\gamma/k\} + \tfrac{1}{2}\right\}\\ &= 2 + \tfrac{1}{k}\Big(\alpha - 1 - \{(n+1)\,\gamma\} - r_{n+1}\Big)\\ &\quad + \left\{\tfrac{1}{k}\{n\,\gamma\} + \tfrac{r_n}{k} + \tfrac{1}{2}\right\}.\end{aligned} \tag{4}$$

Notice that to obtain the above formula, we have only used the fact that n is odd.

1.a) If $\{n\,\gamma\} + r_n < k/2$, then

$$\beta - \{(n+1)\beta\} + \{n\beta\} = 2 + \frac{1}{k}\Big(\underbrace{\alpha - 1 - \{(n+1)\,\gamma\} + \{n\,\gamma\}}_{=0} + r_n - r_{n+1}\Big) + \frac{1}{2}. \tag{5}$$

We have to show that $r_n - r_{n+1} = -k/2$ when $\{n\,\gamma\} + r_n < k/2$ and $\Delta_\alpha(n) = 1$. Since, for k even we have $\lfloor\gamma\rfloor = k/2$, one can notice that

$$(n+1)\,\gamma = q_n k + r_n + \{n\,\gamma\} + \frac{k}{2} + \{\gamma\}.$$

Observe that under the hypothesis $\Delta_\alpha(n)=1$ and k even, we have

$$\{(n+1)\,\gamma\}-\{n\,\gamma\}=\alpha-1=\gamma-k/2=\{\gamma\}. \tag{6}$$

Therefore, $\{(n+1)\,\gamma\}=\{n\,\gamma\}+\{\gamma\}$ and since $r_n+k/2<k$, we conclude that $r_{n+1}=r_n+k/2$ and $\Delta_\beta(n)=2$. In every cases we are dealing with, we always have to check that the quantity r_{n+1} belongs to $\{0,\ldots,k-1\}$.

1.b) If $\{n\,\gamma\}+r_n\geq k/2$, then

$$\beta-\{(n+1)\beta\}+\{n\beta\}=2+\frac{1}{k}\left(\underbrace{\alpha-1-\{(n+1)\,\gamma\}+\{n\,\gamma\}}_{=0}+r_n-r_{n+1}\right)-\frac{1}{2}. \tag{7}$$

Since r_n is an integer and $\{n\gamma\}<1$, we have in this case that $r_n\geq k/2$. Using the same arguments as in the previous case, we obtain

$$(n+1)\,\gamma=q_nk+\frac{k}{2}+r_n+\{(n+1)\,\gamma\}$$

and we conclude that $q_{n+1}=q_n+1$ and $r_{n+1}=r_n-k/2$.

Case 2. Assume both n and k even. We have

$$\begin{aligned}\beta-\{(n+1)\beta\}+\{n\beta\} &= 2+\tfrac{\alpha-1}{k}-\left\{\{(n+1)\,\gamma/k\}+\tfrac{1}{2}\right\}+\{n\,\gamma/k\}\\ &= 2+\tfrac{1}{k}\left(\alpha-1+\{n\,\gamma\}+r_n\right)\\ &\quad -\left\{\tfrac{1}{k}\{(n+1)\,\gamma\}+\tfrac{r_{n+1}}{k}+\tfrac{1}{2}\right\}.\end{aligned} \tag{8}$$

2.a) If $\{(n+1)\,\gamma\}+r_{n+1}<k/2$, then

$$\beta-\{(n+1)\beta\}+\{n\beta\}=2+\frac{1}{k}\left(\underbrace{\alpha-1-\{(n+1)\,\gamma\}+\{n\,\gamma\}}_{=0}+r_n-r_{n+1}\right)-\frac{1}{2}.$$

With the same reasonings as before, we get

$$n\,\gamma=(n+1)\,\gamma-\gamma=q_{n+1}k+r_{n+1}+\{(n+1)\,\gamma\}-\frac{k}{2}-\{\gamma\} \tag{9}$$

and (6) holds. Since $r_{n+1}<k/2$, we deduce that $q_n=q_{n+1}-1$ and $r_n=r_{n+1}+k/2$.

2.b) If $\{(n+1)\,\gamma\}+r_{n+1}\geq k/2$, then as in case 1.b) we know that $r_{n+1}\geq k/2$. Moreover, we have

$$\beta-\{(n+1)\beta\}+\{n\beta\}=2+\frac{1}{k}\left(\underbrace{\alpha-1-\{(n+1)\,\gamma\}+\{n\,\gamma\}}_{=0}+r_n-r_{n+1}\right)+\frac{1}{2}.$$

From (9), we deduce that $r_n=r_{n+1}-k/2$.

Case 3. Assume n and k odd. We have again (4). Observe that a main difference with the first case is that

$$\{(n+1)\,\gamma\}-\{n\,\gamma\}=\{(n+1)\,\alpha\}-\left\{\{n\,\alpha\}+\frac{1}{2}\right\}.$$

3.a) If $\{n\,\gamma\}+r_n<k/2$ and $\{n\alpha\}<1/2$, then (5) becomes here

$$\beta-\{(n+1)\beta\}+\{n\beta\}=2+\frac{1}{k}\left(\underbrace{\alpha-1-\{(n+1)\,\alpha\}+\{n\,\alpha\}}_{=0}+\frac{1}{2}+r_n-r_{n+1}\right)+\frac{1}{2}.$$

Since k is odd, we have $\lfloor\gamma\rfloor = (k+1)/2$ and

$$(n+1)\,\gamma = q_n k + r_n + \{n\,\gamma\} + \frac{k+1}{2} + \{\gamma\}. \tag{10}$$

Observe that under the considered hypothesis, we have

$$\{(n+1)\,\gamma\} - \{n\,\gamma\} = \alpha - 3/2 = \gamma - \frac{k+1}{2} = \{\gamma\}.$$

Since $\{n\alpha\} < 1/2$, then $\{n\gamma\} > 1/2$ and $r_n < (k-1)/2$. We conclude that $r_{n+1} = r_n + (k+1)/2 < k$.

3.b) If $\{n\,\gamma\} + r_n < k/2$ and $\{n\alpha\} > 1/2$, then (5) becomes here

$$\beta - \{(n+1)\beta\} + \{n\beta\} = 2 + \frac{1}{k}\left(-\frac{1}{2} + r_n - r_{n+1}\right) + \frac{1}{2}$$

and

$$\{(n+1)\,\gamma\} - \{n\,\gamma\} = \alpha - 1/2 = \gamma - \frac{k+1}{2} + 1 = \{\gamma\} + 1.$$

Therefore, we have

$$(n+1)\,\gamma = q_n k + r_n + \frac{k-1}{2} + \{(n+1)\,\gamma\} \tag{11}$$

and we get $r_{n+1} = r_n + (k-1)/2 < k$.

3.c) If $\{n\,\gamma\} + r_n \geq k/2$ and $\{n\alpha\} < 1/2$, then (7) becomes here

$$\beta - \{(n+1)\beta\} + \{n\beta\} = 2 + \frac{1}{k}\left(\frac{1}{2} + r_n - r_{n+1}\right) - \frac{1}{2}$$

and (10) holds. Since r_n is an integer and k is odd, we deduce from $\{n\,\gamma\}+r_n \geq k/2$ that $r_n \geq (k-1)/2$. Consequently $r_n+(k+1)/2 \geq k$ and it follows that $q_{n+1} = q_n+1$ and $r_{n+1} = r_n - (k-1)/2$.

3.d) If $\{n\,\gamma\} + r_n \geq k/2$ and $\{n\alpha\} > 1/2$, then (7) becomes here

$$\beta - \{(n+1)\beta\} + \{n\beta\} = 2 + \frac{1}{k}\left(-\frac{1}{2} + r_n - r_{n+1}\right) - \frac{1}{2}$$

and (11) holds. Since $\{n\alpha\} > 1/2$, we have $\{n\gamma\} < 1/2$ and $r_n > (k-1)/2$. Therefore, $r_n + (k-1)/2 > k-1$ and it follows that $q_{n+1} = q_n + 1$ and $r_{n+1} = r_n - (k+1)/2$.

Case 4. Assume n even and k odd. Here (8) holds and one proceeds following the same scheme as in the previous case but here, since n is even, one has to use the fact that

$$\{(n+1)\,\gamma\} - \{n\,\gamma\} = \left\{\{(n+1)\,\alpha\} + \frac{1}{2}\right\} - \{n\,\alpha\}.$$

□

The following result is equivalent to the previous one in the sense that any of these two results implies directly the other one.

Lemma 2. *Let α, β given in* (1) *and $n \geq 1$ be such that $\Delta_\beta(n) = 3$. Then $\Delta_\alpha(n) = 2$.*

Proof. Proceed by contradiction. Assume that $\Delta_\beta(n) = 3$ and that $\Delta_\alpha(n) = 1$. Using the previous lemma, this latter equality implies that $\Delta_\beta(n) = 2$ which is a contradiction. □

In the following two statements, we are considering blocks of consecutive elements of some sequence. These blocks are written using the concatenation of symbols as product operation. Therefore, notation like 2^k means a repetition of k occurrences of the symbol 2.

Remark 2. We can easily show that for any n, the block $\Delta_\alpha(n)\cdots\Delta_\alpha(n+k-1)$ of length k is one of the $k+1$ elements (recall that we are dealing with a Sturmian sequence) of the set

$$\{2^k\}\cup\{2^i12^{k-i-1}\mid i=0,\ldots,k-1\}.$$

Indeed,

$$\begin{aligned} D:=\Delta_\alpha(n)+\cdots+\Delta_\alpha(n+k-1) &= \lfloor (n+k)\,\alpha\rfloor-\lfloor n\,\alpha\rfloor \\ &= k\,\alpha-\{(n+k)\,\alpha\}+\{n\,\alpha\} \end{aligned}$$

and as a function of k, it is easy to check that $k\,\alpha>2k-1$ for all $k\geq 1$ (see Fig. 2). Therefore, the integer D is such that $D>2k-2$ and also $D\leq 2k$, the maximal

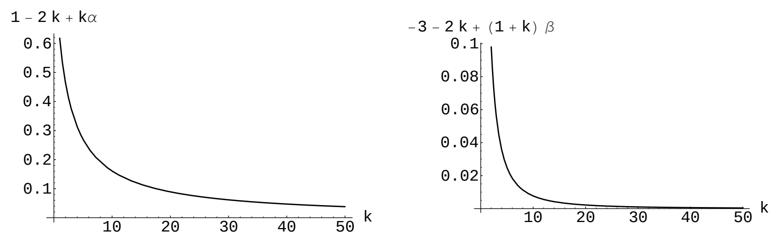

FIGURE 2. The functions $k\,\alpha-2k+1$ and $(k+1)\,\beta-2k-3$.

value being reached in case of the block 2^k. Consequently, for all $n\geq 1$, we have

$$D\in\{2k-1,2k\}.$$

This means that any block of length k contains at most one 1. From this, one can also deduce that any block of length $k+1$ in $(\Delta_\alpha(n))_{n\geq 1}$ contains $0,1$ or 2 occurrences of 1.

In the same way, for all $n\geq 1$, the block $\Delta_\beta(n)\cdots\Delta_\beta(n+k-1)$ of length k is one of the $k+1$ elements in

$$\{2^k\}\cup\{2^i32^{k-i-1}\mid i=0,\ldots,k-1\}. \tag{12}$$

Moreover, 2^{k+1} is not a block of length $k+1$ occurring in the sequence and $32^{k-1}3$ is the only block of length $k+1$ containing two occurrences of 3. Indeed, assume that $\Delta_\beta(n)+\cdots+\Delta_\beta(n+k)=2^{k+1}$. This implies

$$(k+1)\beta=2k+2+\{(n+k+1)\,\beta\}-\{n\,\beta\}\leq 2k+3.$$

But as a function of k, we have $(k+1)\beta>2k+3$ (see Fig. 2). This proves that the block 2^{k+1} does not occur. Moreover, any block of length $k+1$ containing two occurrences of 3 in another configuration than $32^{k-1}3$ would lead to a block of length k with two occurrences of 3.

Lemma 3. *Let α, β given in* (1) *and $k\geq 2$. If $\Delta_\alpha(n)\cdots\Delta_\alpha(n+k-1)=2^k$, then*

$$\Delta_\beta(n)\cdots\Delta_\beta(n+k-1)\in\{2^i32^{k-i-1}\mid i=0,\ldots,k-1\}.$$

In particular, if $\Delta_\alpha(n)\cdots\Delta_\alpha(n+k-1)=2^k$ *and* $\Delta_\beta(n)\cdots\Delta_\beta(n+k-2)=2^{k-1}$, *then* $\Delta_\beta(n+k-1)=3$.

Proof. We set

$$A := \frac{1}{k}\left(k\,\alpha-\{(n+k)\,\alpha\}+\{n\,\alpha\}\right)$$

and

$$B := \alpha-\left\{(n+k)\,\frac{\alpha-1}{k}\right\}+\left\{n\,\frac{\alpha-1}{k}\right\}.$$

The first assumption can be written $kA=\lfloor(n+k)\,\alpha\rfloor-\lfloor n\,\alpha\rfloor=2k$, i.e., $A=2$, and in view of (12), we have to show that $\lfloor(n+k)\,\beta\rfloor-\lfloor n\,\beta\rfloor=2k+1$. Using (3), we get

$$\lfloor(n+k)\,\beta\rfloor-\lfloor n\,\beta\rfloor=2k-1+B.$$

In particular, this implies that B is an integer. To conclude the proof, it is enough to show that for all $n\ge 1$,

$$A-B\in\{-\frac{1}{k},0,1-\frac{1}{k}\}.$$

If this can be shown, then whenever $A=2$, B being an integer and for $k\ge 2$, $-\frac{1}{k}$ and $1-\frac{1}{k}$ being not integer, we must have $A=B$ and $\lfloor(n+k)\,\beta\rfloor-\lfloor n\,\beta\rfloor=2k+1$. We have

$$A-B=\left\{(n+k)\,\frac{\alpha-1}{k}\right\}-\left\{n\,\frac{\alpha-1}{k}\right\}-\frac{1}{k}\{(n+k)\,\alpha\}+\frac{1}{k}\{n\,\alpha\}.$$

For any $m\ge 1$, we consider the Euclidian division of $\lfloor m\,(\alpha-1)\rfloor$ by k:

$$m\,(\alpha-1)=q_m\,k+r_m+\{m\,(\alpha-1)\}$$

to define $q_m\in\mathbb{N}$ and $r_m\in\{0,\dots,k-1\}$. Therefore, for all $m\ge 1$,

$$\left\{m\,\frac{\alpha-1}{k}\right\}=\frac{1}{k}\{m\,\alpha\}+\frac{r_m}{k}. \tag{13}$$

Now, observe that

$$(n+k)\,(\alpha-1)=(q_n+\lfloor\alpha-1\rfloor)\,k+r_n+\{n\,(\alpha-1)\}+k\,\{\alpha-1\}$$

and we have to study $k\{\alpha-1\}=k\{\gamma-k/2\}$ using the definition (2). Since $\lfloor\gamma\rfloor=\lceil k/2\rceil$, for k even, we have

$$k\{\gamma-k/2\}=k\{\gamma\}=k(\gamma-\lfloor\gamma\rfloor)=k\frac{\sqrt{k^2+4k}-k}{2}$$

and for k odd, since $\lfloor\gamma\rfloor=(k-1)/2$ and $\lfloor\gamma+1/2\rfloor=(k+1)/2$, we get

$$k\{\gamma-k/2\}=k\{\gamma+1/2\}=k(\gamma+1/2-\lfloor\gamma+1/2\rfloor)=k(\gamma-k/2)=k\frac{\sqrt{k^2+4k}-k}{2}.$$

In both cases, we have the same function $f(k)$ which can easily be seen (see Fig. 3 for a sketch of $f(k)-k+1$) to satisfy

$$k-1<k\{\gamma-k/2\}<k.$$

So $k\{\alpha-1\}=k-1+\{k\{\alpha-1\}\}$ and

$$(n+k)\,(\alpha-1)=(q_n+\lfloor\alpha-1\rfloor)k+r_n+k-1+\underbrace{\{n\,(\alpha-1)\}+\{k\{\alpha-1\}\}}_{:=C}.$$

The conclusion follows easily. We have three cases to consider

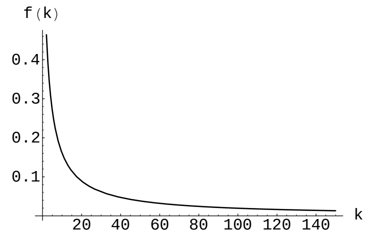

FIGURE 3. $f(k) - k + 1 > 0$.

- If $C \in [0,1)$ and $r_n = 0$, then $q_{n+k} = q_n + \lfloor \alpha - 1 \rfloor$ and $r_{n+k} = k - 1$.
- If $C \in [0,1)$ and $r_n \in \{1, \ldots, k-1\}$, then $q_{n+k} = q_n + \lfloor \alpha - 1 \rfloor + 1$ and $r_{n+k} = r_n - 1$.
- If $C \in [1,2)$, then $q_{n+k} = q_n + \lfloor \alpha - 1 \rfloor + 1$ and $r_{n+k} = r_n$.

Consequently, using (13), $A - B$ reduces to $(r_{n+k} - r_n)/k \in \{-1/k, 0, 1 - 1/k\}$. $\square$

Lemma 4. *Let α, β given in* (1) *and $k \geq 1$. If $\Delta_\beta(n) \cdots \Delta_\beta(n+k) = 32^{k-1}3$, then $\Delta_\alpha(n) \cdots \Delta_\alpha(n+k) = 2^{k+1}$.*

Proof. We assume that

$$k \lfloor (n+k+1)\, \beta \rfloor - k \lfloor n\, \beta \rfloor = 2k(k+2). \tag{14}$$

We proceed by contradiction and assume that $\Delta_\alpha(n) \cdots \Delta_\alpha(n+k)$ contains 1 or 2 occurrences of 1 (from Remark 2, these are the only cases to consider), i.e.,

$$\lfloor (n+k+1)\, \alpha \rfloor - \lfloor n\, \alpha \rfloor \in \{2k, 2k+1\}.$$

Using (3), $\lfloor (n+k+1)\, \alpha \rfloor - \lfloor n\, \alpha \rfloor$ is written

$$(k+1)(k\beta - 2k + 1) - \{(n+k+1)k\beta\} + \{nk\beta\} \in \{2k, 2k+1\}. \tag{15}$$

Set $\beta' = k\beta$ and subtract (15) from (14). We have that

$$\begin{aligned} &(k+1)\beta' - k\{(n+k+1)\beta'/k\} + k\{n\beta'/k\} \\ &\quad - \left[(k+1)\beta' - \{(n+k+1)\beta'\} + \{n\beta'\} \right] \quad \in \{k, k+1\} \end{aligned}$$

Proceeding as in the proof of the previous lemma, one can show that this expression divided by k:

$$\frac{1}{k}(\{(n+k+1)\beta'\} - \{n\beta'\}) - \{(n+k+1)\beta'/k\} + \{n\beta'/k\}$$

belongs to $\{-\frac{1}{k}, 0, 1 - \frac{1}{k}\}$ by noticing that $\lfloor \beta' \rfloor = 2k$ and $\{\beta'\} > k/(k+1)$. But this expression divided by k must belong to $\{1, 1 + \frac{1}{k}\}$ leading to a contradiction. $\square$

2. An invariant game

For all $k \geq 1$, let us present a variation of Wythoff's game that we call $G(\alpha_k)$, this notation will become clear in a few lines (see Theorem 4). The set of moves of the invariant game $G(\alpha_k)$ is

$$\mathcal{M}_W \setminus \{(2i, 2i) \mid 0 < i < k\} \cup \{(2k+1, 2k+2), (2k+2, 2k+1)\}$$

where $\mathcal{M}_W$ is the set of Wythoff's moves. In other terms, this game can be described as follows: either take a positive number from a single pile, or take $(2i, 2i), i < k$

from both, or $2k+1, k>0$ from one and $2k+2$ from the other. The first player unable to move loses. Note that the rules of the game $G(\alpha_k)$ are due to computer experiments, so as to fit the Beatty sequence $(\lfloor n\alpha_k\rfloor, \lfloor n\beta_k\rfloor)$, as it will be shown in Theorem 4. Since the moves are symmetric on the two piles of tokens, we can restrict ourselves to positions (x,y) with $x\le y$. Recall that k was given once and for all, thus the P-positions will be denoted (A_n, B_n). For instance, the first ones in the case $k=2$ are

n	0	1	2	3	4	5	6	7	8	...
A_n	0	1	3	5	6	8	10	12	13	...
B_n	0	2	4	7	9	11	14	16	18	...

Remark 3. Observe that for $k=1$, we have $\mathcal{M}_W \cup \{(3,4),(4,3)\}$ as set of moves for $G(\alpha_1)$ and it is shown in [4] that adding such a move to $\mathcal{M}_W$ does not change the set of P-positions of Wythoff's game. So the game $G(\alpha_1)$ has exactly the same set of P-positions as the classical Wythoff's game. Note that in [4], it is also proved that adding any move of the form $(k,k+1)$ with $k\neq 1,2$ does not change the set of P-positions of Wythoff's game.

Definition 3. Let $k\ge 2$. We define recursively a sequence $(A_n,B_n)_{n\ge 0}$ as follows.

$$(A_0,B_0),(A_1,B_1),\ldots,(A_k,B_k)=(0,0),(1,2),(3,4),\ldots,(2k-1,2k),$$

$$A_n = \mathrm{Mex}\{A_i, B_i \mid i<n\}.$$

For all $n\ge k$, if the following condition holds true

$$\begin{aligned} & A_{n+1}-A_n = 2 \\ \wedge\quad & \Big[(B_n-A_n = B_{n-k+1}-A_{n-k+1}+1 \wedge A_{n+1}-A_{n-k}\neq 2k+1) \\ & \qquad \vee\ B_{n-k}-A_{n-k}\neq B_n-A_n-1\Big] \end{aligned}$$

then $B_{n+1}-A_{n+1}=B_n-A_n$, otherwise $B_{n+1}-A_{n+1}=B_n-A_n+1$.

The sequence $(A_n)_{n\ge 0}$ is obviously increasing and the sequence $(B_n-A_n)_{n\ge 0}$ is non-decreasing. Therefore the sequence $(B_n)_{n\ge0}$ is also increasing. By the Mex rule defining $(A_n)_{n\ge0}$, we assert that $\{A_n \mid n\ge 1\}$ and $\{B_n\mid n\ge 1\}$ make a partition of $\mathbb{N}_{\ge 1}$.

Lemma 5. *Let $k\ge 2$ (the same as in Definition 3). For the sequence $(A_n,B_n)_{n\ge0}$ given in Definition 3, we have $A_{n+1}-A_n\in\{1,2\}$ for all $n\ge 0$.*

Proof. We proceed by induction on $n\ge 0$. We have $A_{n+1}-A_n\in\{1,2\}$ for all $n<k$. Assume now that $A_{j+1}-A_j\in\{1,2\}$ for all $j\le n$. The definition of the sequence implies that, for all $j\le n$,

$$B_{j+1}-B_j\in\{2,3\}. \tag{16}$$

If $A_{n+1}=A_n+1$, then the result holds. Otherwise, there exists $i\le n$ such that $A_n+1=B_i$. From (16), we deduce that $B_{i+1}\ge A_n+3$ and from the Mex rule defining $(A_n)_{n\ge0}$, we conclude that $A_{n+1}=A_n+2$. □

The next corollary follows directly from the definition of the sequence and the above lemma.

Corollary 2. *Let $k \geq 2$ (the same as in Definition 3). For the sequence $(A_n, B_n)_{n\geq 0}$ given in Definition 3, we have, for all $n \geq 0$,*

$$(A_{n+1} - A_n, B_{n+1} - B_n) \in \{(1, 2), (2, 2), (2, 3)\}.$$

The next result is correlated with the observation made in Remark 2.

Lemma 6. *Let $k \geq 2$ (the same as in Definition 3). For the sequence $(A_n, B_n)_{n\geq 0}$ given in Definition 3, any subsequence of k consecutive differences of the kind*

$$(A_{i+1} - A_i, B_{i+1} - B_i), \ldots, (A_{i+k} - A_{i+k-1}, B_{i+k} - B_{i+k-1})$$

contains at most one occurrence of $(2, 3)$.

Proof. Assume to the contrary that there exists $n \geq 0$ and $i \in \{1, \ldots, k-1\}$ such that

$$(A_{n+i+1} - A_{n+i}, B_{n+i+1} - B_{n+i}) = (A_{n+k+1} - A_{n+k}, B_{n+k+1} - B_{n+k}) = (2, 3).$$

In particular, this means that

$$B_{n+k+1} - A_{n+k+1} = B_{n+k} - A_{n+k} + 1 \geq B_{n+i+1} - A_{n+i+1} + 1 = B_{n+i} - A_{n+i} + 2.$$

By the definition of the sequence $(A_n, B_n)_{n\geq 0}$, since $(A_{n+k+1} - A_{n+k}, B_{n+k+1} - B_{n+k}) = (2, 3)$, the following 3-terms condition should be satisfied

$$\begin{array}{cc} & (B_{n+k} - A_{n+k} \neq B_{n+1} - A_{n+1} + 1 \vee A_{n+k+1} - A_n = 2k + 1) \\ \wedge & B_n - A_n = B_{n+k} - A_{n+k} - 1. \end{array}$$

The last term in this condition should be satisfied. Therefore all the pairs of differences $(A_{n+1} - A_n, B_{n+1} - B_n), \ldots, (A_{n+k} - A_{n+k-1}, B_{n+k} - B_{n+k-1})$ are equal to $(2, 2)$ except for $(A_{n+i+1} - A_{n+i}, B_{n+i+1} - B_{n+i})$ which is equal to $(2, 3)$. Since $A_{n+k+1} - A_{n+k} = 2$, we conclude that $A_{n+k+1} - A_n$ is even and therefore $A_{n+k+1} - A_n \neq 2k + 1$. From the above discussion, we can also conclude that $B_{n+k} - A_{n+k} = B_n - A_n + 1$. Therefore the first two terms of the condition are never satisfied which is a contradiction. □

Lemma 7. *Let $k \geq 2$ (the same as in Definition 3). For the sequence $(A_n, B_n)_{n\geq 0}$ given in Definition 3, any subsequence of $k + 1$ consecutive differences*

$$(A_{i+1} - A_i, B_{i+1} - B_i), \ldots, (A_{i+k+1} - A_{i+k}, B_{i+k+1} - B_{i+k})$$

contains at most one occurrence of $(1, 2)$.

Proof. Assume that $A_n = j$ and $A_{n+1} = j + 1$ for some $n, j \geq 2$ (this means that $A_{n+1} - A_n = 1$). Let us show that the next k differences $A_{n+2} - A_{n+1}, \ldots, A_{n+k+1} - A_{n+k}$ are all equal to 2. Observe that there exists m such that $j - 1 = B_m$ and $j + 2 = B_{m+1}$ because otherwise, we would have $B_{m+1} - B_m > 3$ which contradicts Corollary 2. In particular, $B_{m+1} - B_m = 3$. By Lemma 6, we conclude that

$$B_{m+2}, \ldots, B_{m+k} = j + 4, \ldots, j + 2k$$

and since $B_{m+k+1} - B_{m+k} \geq 2$, we also have $B_{m+k+1} \geq j + 2k + 2$. From the Mex rule defining the sequence, we get that

$$A_{n+2}, \ldots, A_{n+k}, A_{n+k+1} = j + 3, \ldots, j + 2k - 1, j + 2k + 1.$$

□

Lemma 8. *Let $k \geq 2$ (the same as in Definition 3). For the sequence $(A_n, B_n)_{n\geq 0}$ given in Definition 3, any subsequence of $k+1$ consecutive differences*

$$(A_{i+1} - A_i, B_{i+1} - B_i), \dots, (A_{i+k+1} - A_{i+k}, B_{i+k+1} - B_{i+k})$$

contains at most two elements in $\{(1,2),(2,3)\}$.

Proof. Assume to the contrary that there exist $i \in \{1, \dots, k-2\}$ and $j \in \{2, \dots, k-1\}$ such that $i < j$ and

$$\begin{cases} B_{n+i+1} - A_{n+i+1} &= B_{n+i} - A_{n+i} + 1 \\ B_{n+j+1} - A_{n+j+1} &= B_{n+j} - A_{n+j} + 1 \\ B_{n+k+1} - A_{n+k+1} &= B_{n+k} - A_{n+k} + 1. \end{cases}$$

Observe that if $(A_{n+k+1} - A_{n+k}, B_{n+k+1} - B_{n+k}) = (2,3)$, then by Definition 3, we should have $B_n - A_n = B_{n+k} - A_{n+k} - 1$ which is not the case. Consequently, $(A_{n+k+1} - A_{n+k}, B_{n+k+1} - B_{n+k}) = (1,2)$. By Lemma 7, we get that $(A_{n+j+1} - A_{n+j}, B_{n+j+1} - B_{n+j}) = (2,3)$. By Lemma 6, we then get that $(A_{n+i+1} - A_{n+i}, B_{n+i+1} - B_{n+i}) = (1,2)$. But these two occurrences of $(1,2)$ contradict Lemma 7. □

Lemma 9. *Let $k \geq 2$ (the same as in Definition 3). For the sequence $(A_n, B_n)_{n\geq 0}$ given in Definition 3, any subsequence of k consecutive differences*

$$(A_{i+1} - A_i, B_{i+1} - B_i), \dots, (A_{i+k} - A_{i+k-1}, B_{i+k} - B_{i+k-1})$$

contains at least one element in $\{(1,2),(2,3)\}$.

Proof. Assume to the contrary that there exists $n \geq 0$ such that

$$(A_{n+1} - A_n, B_{n+1} - B_n) = (A_{n+2} - A_{n+1}, B_{n+2} - B_{n+1}) = \dots$$

$$= (A_{n+k} - A_{n+k-1}, B_{n+k} - B_{n+k-1}) = (2,2).$$

We choose the smallest possible such n. Hence $A_{n+k} - A_n = 2k$. According to Definition 3, we should have either $B_{n+k-1} - A_{n+k-1} = B_n - A_n + 1$ or $B_{n+k-1} - A_{n+k-1} - 1 \neq B_{n-1} - A_{n-1}$. The first condition is false since $B_{n+k-1} - A_{n+k-1} = B_n - A_n$. And by minimality of n, we have $B_{n-1} - A_{n-1} = B_n - A_n - 1$, contradicting the second condition. □

Theorem 3 (Recursive characterization)**.** *The P-positions of $G(\alpha_k)$, $k \geq 2$, are given by the sequence $(A_n, B_n)_{n\geq 0}$ in Definition 3.*

Proof. We first show that there is no move from a position (A_n, B_n) to some position (A_m, B_m) with $0 \leq m < n$. Assume that such a move exists. Then this move is necessarily of the form $A_n \to A_m$ and $B_n \to B_m$. Indeed, if $A_n \to B_m$ and $B_n \to A_m$, then we have $A_m < B_m < A_n < B_n$. Hence $0 < A_n - B_m < B_n - A_m - 1$, and no rule of $G(\alpha_k)$ allows to play a move (x, y) with $|x - y| > 1$ and $xy \neq 0$. Notice that, since the four numbers A_n, A_m, B_n, B_m are pairwise distinct, the only moves to be considered are those played on both piles. We now consider the three possible cases about n and m:

- $m < n \leq k$. All the differences $(A_n - A_m, B_n - B_m)$ are of the form $(2i, 2i)$ with $0 < i < k$, which are forbidden moves according to the rules of $G(\alpha_k)$.

- $m \leq k < n$. Since $B_n - A_n \geq 2 > B_m - A_m$, the only possible move is such that $(A_n - A_m, B_n - B_m)$ is equal to $(2k+1, 2k+2)$ or $(2k+2, 2k+1)$. If the latter case occurs, we would have $B_m - A_m > B_n - A_n$. This is impossible because the sequence $(B_i - A_i)_{i\geq 0}$ is non-decreasing. Assume that $(A_n - A_m, B_n - B_m) = (2k+1, 2k+2)$. Since $B_m - A_m < 2$, this implies $B_n - A_n = 2$. In the sequence $(A_i, B_i)_{i\geq 0}$ there is a unique pair (A_n, B_n) satisfying $B_n - A_n = 2$. It is $(A_{k+1}, B_{k+1}) = (2k+1, 2k+3)$ because $(A_{k+2}, B_{k+2}) = (2k+2, 2k+5)$. Therefore, playing the move $(2k+1, 2k+2)$ from (A_{k+1}, B_{k+1}) leads to $(0, 1)$ which is not in the sequence $(A_i, B_i)_{i\geq 0}$.
- $k < m < n$. Assume that the move $(A_n - A_m, B_n - B_m)$ is of the form (x, x) with $x \neq 2i$, for all $0 < i < k$. Hence $B_m - A_m = B_n - A_n$. According to Corollary 2, this implies that

$$A_n - A_m = B_n - B_m = 2(n - m).$$

  Hence $n - m \geq k$ and the $n - m$ consecutive differences $(A_{m+1} - A_m, B_{m+1} - B_m), \dots, (A_n - A_{n-1}, B_n - B_{n-1})$ are equal to $(2, 2)$. This contradicts Lemma 9. Now assume that the move $(A_n - A_m, B_n - B_m)$ is of the form $(2k+1, 2k+2)$. Then according to Corollary 2, all the $k+1$ pairs of differences $(A_{m+1} - A_m, B_{m+1} - B_m), \dots, (A_n - A_{n-1}, B_n - B_{n-1})$ are equal to $(2, 2)$ except for one of them $(A_i - A_{i-1}, B_i - B_{i-1})$, which is equal to $(1, 2)$. By Lemma 9, we deduce that $m + 1 < i < n$. Hence we have that $(A_n - A_{n-1}, B_n - B_{n-1}) = (2, 2)$. Notice that $B_{n-k-1} - A_{n-k-1} = B_{n-1} - A_{n-1} - 1$ and $A_n - A_{n-k-1} = 2k+1$, which contradicts Definition 3. As in the previous case, one can check that the case $(A_n - A_m, B_n - B_m) = (2k+2, 2k+1)$ does not occur.

Let (a, b) be a game position which is not in the sequence $(A_n, B_n)_{n\geq 0}$. We now show that it is always possible to play from (a, b) to a position in $(A_n, B_n)_{n\geq 0}$.

Without loss of generality, assume that $a \leq b$. If $a = 0$, then we can play from (a, b) to $(0, 0)$. Now assume that $a > 0$. Since $(A_n, B_n)_{n\geq 1}$ makes a partition of $\mathbb{N}_{\geq 1}$, there exists $n > 0$ such that $a = A_n$ or $a = B_n$. If $a = B_n$, then we play $b \to A_n$ and leave the other pile unchanged. If $a = A_n$, then we consider three cases:

- $b > B_n$. Then play $b \to B_n$ and leave the other pile unchanged.
- $b < B_n$ and there exists $0 < i < n$ such that $b = B_i$. Then play $a \to A_i$ and leave the other pile unchanged.
- $b < B_n$ and there exists $j > n$ such that $b = A_j$. Since $A_j - A_n < B_n - A_n$, there exists $i < n$ such that $B_i - A_i = A_j - A_n$. We choose the smallest i having this property. If $A_n - A_i \neq 2p$ for all $0 < p < k$, then we play $A_n \to A_i$ and $A_j \to B_i$. Otherwise, there exists $0 < p < k$ such that $A_n - A_i = A_j - B_i = 2p$. According to Corollary 2 and Lemma 7, the $n - i$ differences $A_n - A_{n-1}, \dots, A_{i+1} - A_i$ equal 2 and $n - i = p < k$. Since $B_n - A_n > B_i - A_i$ and by Lemma 6, we have $(A_{i+1} - A_i, B_{i+1} - B_i), \dots, (A_n - A_{n-1}, B_n - B_{n-1})$ are equal to $(2, 2)$ except for one of them. There exists a unique $t \in \{i+1, \dots, n\}$ such that $(A_t - A_{t-1}, B_t - B_{t-1}) = (2, 3)$. Now, by minimality of i, we have that $(A_i - A_{i-1}, B_i - B_{i-1}) \in \{(1, 2), (2, 3)\}$, and by Lemma 6, $(A_i - A_{i-1}, B_i - B_{i-1}) = (1, 2)$. According to Lemma 8, all the differences $(A_{t-k} - A_{t-k-1}, B_{t-k} - B_{t-k-1}), \dots, (A_{i-1} - A_{i-2}, B_{i-1} - B_{i-2})$ are equal to $(2, 2)$. Notice that $t - k \leq n - k < i$. We can conclude

that $A_n - A_{n-k} = 2k+1$ and $A_j - A_{n-k} = 2k+2$. Hence we can play $(A_n, A_j) \to (A_{n-k}, B_{n-k})$.

□

Theorem 4 (Algebraic characterization)**.** *Let* $\alpha := \alpha_k$, $\beta := \beta_k$ *given in* (1) *for some* $k \geq 2$. *If for all* $n \geq 0$, *we set* $(A'_n, B'_n) := (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, *then the set of P-positions of* $G(\alpha_k)$ *is exactly* $\{(A'_n, B'_n) \mid n \geq 0\}$.

Proof. We simply have to show that the sequence $(A'_n, B'_n)_{n\geq 0}$ is verifying the recursive characterization given in the previous theorem.

A. We first check the initial conditions. It is obvious that $(A'_0, B'_0) = (0,0)$ and $(A'_1, A'_2) = (1,2)$, see (1). It is enough to check that $\Delta_\alpha(i) = \Delta_\beta(i) = 2$ for $i = 1, \ldots, k-1$. Since $\Delta_\alpha(0) = 1$ and $\Delta_\beta(0) = 2$, it is the same to verify that $\Delta_\alpha(0) + \cdots + \Delta_\alpha(k-1) = \lfloor k\,\alpha \rfloor = 2k-1$ and $\lfloor k\,\beta \rfloor = 2k$. As a function of k, one can easily check that $2k-1 < k\alpha < 2k$ (see Fig. 2) and in the same way, $2k < k\beta < 2k+1$.

B. As it was already observed by A. Fraenkel in [9], the facts that the sequences $(A'_n)_{n\geq 1}$ and $(B'_n)_{n\geq 1}$ are complementary Beatty increasing sequences and that $A'_n < B'_n$ for all $n \geq 1$ imply that we necessarily have $A'_n = \text{Mex}\{A'_i, B'_i \mid i < n\}$.

C. Let $n \geq k$. We now turn our attention to the determination of $B'_{n+1} - A'_{n+1}$ with respect to the value of $B'_n - A'_n$.

C.1 Assume that the following conditions

(**H1**) $A'_{n+1} - A'_n = 2$,
(**H2**) $B'_n - A'_n = B'_{n-k+1} - A'_{n-k+1} + 1$,
(**H3**) $A'_{n+1} - A'_{n-k} \neq 2k+1$

are satisfied. We have to show that $B'_{n+1} - A'_{n+1} = B'_n - A'_n$. From (H1) and (H3), we get $\lfloor n\alpha \rfloor - \lfloor (n-k)\,\alpha \rfloor \neq 2k-1$ so from Remark 2, we deduce that $\lfloor n\alpha \rfloor - \lfloor (n-k)\,\alpha \rfloor = 2k$. Consequently, $\Delta_\alpha(n-i) = \lfloor (n-i+1)\alpha \rfloor - \lfloor (n-i)\alpha \rfloor = 2$ for $i = 0, \ldots, k$. Using Lemma 3, we know that the two overlapping blocks $x := \Delta_\beta(n-k) \cdots \Delta_\beta(n-1)$ and $y := \Delta_\beta(n-k+1) \cdots \Delta_\beta(n)$ both contain exactly one occurrence of 3. This is not enough to conclude, at this stage it could be possible that $\Delta_\beta(n-k) \cdots \Delta_\beta(n)$ is equal to either

$$32^{k-1}3 \text{ or } 2^i 3 2^{k-i}, \text{ for some } i \in \{1, \ldots, k-1\}. \tag{17}$$

But from (H2), we get $\lfloor n\beta \rfloor - \lfloor (n-k+1)\,\beta \rfloor = 2k-1$ and $\Delta_\beta(n-k+1) \cdots \Delta_\beta(n-1)$ contains the only 3 occurring in x and y. Consequently, $\Delta_\beta(n) = 2$ which means that $B'_{n+1} - A'_{n+1} = B'_n - A'_n$. Notice that the remaining condition occurring in the recursive definition of Theorem 3:

(**H4**) $B'_{n-k} - A'_{n-k} \neq B'_n - A'_n - 1$

has not been considered in our discussion. It is never satisfied whenever (H1) and (H3) are satisfied. Indeed, assume that (H1), (H3), (H4) are satisfied. We show that such a situation never occurs. With (H1) and (H3), we know that $\Delta_\alpha(n-i) = 2$ for $i = 0, \ldots, k$ and that we have (17). (H4) gives $B'_n - B'_{n-k} \neq 2k+1$ which means that $\Delta_\beta(n-k) \cdots \Delta_\beta(n-1) = 2^k$ contradicting (17).

If condition (Hi) is not satisfied, $i = 1,2,3,4$, we write ($\neg$Hi) as a shorthand. Therefore (H1), (H2), (H3) and ($\neg$H4) are *incompatible*: it is impossible to have simultaneously (H1), (H2), (H3) and ($\neg$H4). In all the other cases that we consider below to cover all the situations, we do not pay attention to some of the four conditions. In fact, it should be understood that *if the conditions are compatible*, then we show how to get the expected thesis. In case of incompatible conditions, there is nothing to show. Table 1 enumerate all the possible situations and give the corresponding case where we are considering it. Each of the conditions (H1) to (H4) can be either True or False.

C.	1	1	2	4	5	5	2	4	3	3	3	3	3	3	3	3
H1	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
H2	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
H3	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
H4	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

TABLE 1. The possible truth values and the corresponding case.

C.2 Assume now that we have (H1), ($\neg$H3) and (H4). We have to show that $B'_{n+1}-A'_{n+1} = B'_n-A'_n$, i.e., $\Delta_\beta(n) = 2$. From (H1) and ($\neg$H3), we get $A'_n-A'_{n-k} = 2k-1$ and $\Delta_\alpha(n-k)\cdots\Delta_\alpha(n-1)$ contains exactly one occurrence of 1. (H4) can be written as $B'_n - B'_{n-k} \neq 2k$. Therefore $\Delta_\beta(n-k)\cdots\Delta_\beta(n-1)$ contains exactly one occurrence of 3, say $\Delta_\beta(n-j) = 3$ for some $j \in \{1,\dots,k\}$. If $j < k$, from Remark 2, $\Delta_\beta(n-k+1)\cdots\Delta_\beta(n)$ contains at most one occurrence of 3, we deduce that $\Delta_\beta(n) = 2$. If $j = k$, $\Delta_\beta(n-k) = 3$ and we proceed by contradiction. Assume that $\Delta_\beta(n) = 3$, so from Lemma 4, we deduce that $\Delta_\alpha(n-k)\cdots\Delta_\alpha(n) = 2^{k+1}$ which is a contradiction (we know that this block contains an occurrence of 1).

C.3 Assume now that (H1) is not satisfied. This means that $\Delta_\alpha(n) = 1$ and we have to show that $B'_{n+1} - A'_{n+1} = B'_n - A'_n + 1$, i.e., that $\Delta_\beta(n) = \Delta_\alpha(n) + 1 = 2$. It is an immediate consequence of Lemma 1.

C.4 Assume now that we have (H1), ($\neg$H3) and ($\neg$H4). We have to show that $B'_{n+1}-A'_{n+1} = B'_n-A'_n+1$, i.e., that $\Delta_\beta(n) = 3$. As in the second case, from (H1) and ($\neg$H3), $\Delta_\alpha(n-k)\cdots\Delta_\alpha(n-1)$ contains exactly one occurrence of 1. From ($\neg$H4), we get $\Delta_\beta(n-k)\cdots\Delta_\beta(n-1) = 2^k$. Proceed by contradiction and assume that $\Delta_\beta(n) = 2$. Therefore, we would have a block of length $k+1$ of the form 2^{k+1} in $(\Delta_\beta(n))_{n\geq 1}$. This is impossible in view of Remark 2.

C.5 The last case not yet considered is when we have (H1), ($\neg$H2), (H3) (and as a consequence of (H1) and (H3), we get ($\neg$H4)). As in the first case, we have that $\Delta_\alpha(n-i) = 2$ for all $i = 0,\dots,k$ and (17) also holds: $\Delta_\beta(n-k)\cdots\Delta_\beta(n)$ is equal to either $32^{k-1}3$ or 2^i32^{k-i}, for some $i \in \{1,\dots,k-1\}$. But from ($\neg$H2), we get $\lfloor n\beta\rfloor - \lfloor (n-k+1)\,\beta\rfloor \neq 2k-1$ so, $\Delta_\beta(n-k+1)\cdots\Delta_\beta(n-1) = 2^{k-1}$ and therefore $\Delta_\beta(n-k) = \Delta_\beta(n) = 3$. This concludes the last case: $\Delta_\beta(n) = \Delta_\alpha(n) + 1$. $\square$

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