A deletion game on graphs : "Le Pic'arete"

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Abstract

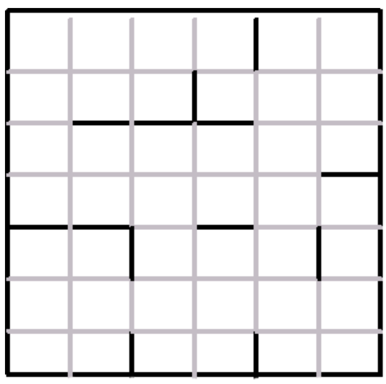
This paper presents a deletion game on graphs, called "Le Pic'arete". Two players alternately remove an edge of a given graph, one player getting one point each time the deletion of an edge isolates a vertex (so that you can get at best two points by edge deleted). When no edge remains, the player who has the maximum number of points is declared the winner. For instance, if we consider this game on the grid graph, we obtain a variant of famous "Dots and Boxes" game.

Our study does not consist in finding the Grundy's function of this game (for a change), but trying to make an estimate of the value of an initial game configuration, i.e. the difference between the maxima numbers of points attainable by both players. Finding a winning strategy will lead us to define a classification of game configurations, according to the parity of their numbers of edges, vertices and sizes of hanging trees.

1 Presentation

The original game (or how fighting against student's boredom)...

Time spent on halls chairs often seems too long...following game could help students killing it: find a sheet of squared paper and define a grid on it (the game zone) by darkening segments (generally we draw a rectangular outline, and darken several segments inside).



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Then two players alternately darken one segment of the grid. Each player gets one point when darkening the fourth side of a square. The game ends when all the segments have been darkened, the winner being the player with the highest score.
For having played this game for several years, the natural occuring question is the following : "Does it exist a strategy that leads me to victory, or at least, that helps me getting the best score. And in that case, which one ?" Actually this game can be considered as a variant of the "Dots and Boxes" game (studied by Berlekamp in [4]). In this game, players are allowed to move again after having won points.

and its generalization : a deletion game on graphs

We consider in this paper a generalization of this student game: we play on a simple graph with no loop (in our case, it could be the dual of the initial grid). Both players remove alternately an edge, one player getting one point each time the deletion of an edge isolates a vertex (so that you can get at best two points by edge deleted). The goal remains the same, i.e. getting the most points.
Another deletion games on graphs have ever been studied before. In [1] for exemple, the game is played on an hypergraph, both players deleting either a vertex or an edge. We can consider the game "Geography" too (in [2],[3]), where the deletion of an edge (or a vertex) depends on the edge (or vertex) deleted before. Games on graphs are often defined with the rule "last player that moves wins". That is the case in exemples cited before, unlike "Le Pic'arete", where the winner is defined by the number of points he gets. In terms of winning strategy, this consists in finding the maximum number of points attainable by both players (which will naturally decide whether the initial configuration is a first or a second-player win).

2 Classifying the configurations

Let C be a game configuration, we call g_1 and g_2 the maximal scores of the first and second players.
We define the **value** of C, $w(C)$ as following:

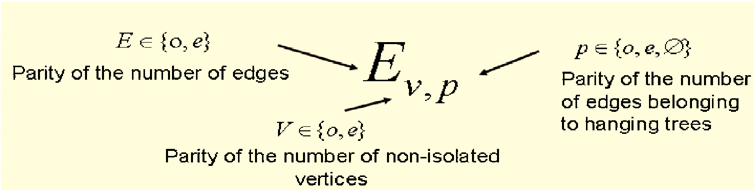
$$w(C) = g_1 - g_2$$

i.e. $w(C)$ is the difference between the maxima numbers of points attainable by both players. This means that a configuration with a positive value is a first-player win. In a zero-value configuration, no matter who begins, as no one has a winning strategy.
As exemple, if you play on a graph such as a triangle, you should begin as this configuration has a value equal to 1. On the other hand, a square is a second-player win, as its value equals -2.

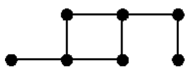
The problem now consists in finding which parameters of a given configuration play a role in the estimate of its value. Following parameters have been considered according to values of small configurations :

- number of edges
- number of non-isolated vertices
- number of edges belonging to hanging trees
- number of connected graphs constituting the configuration

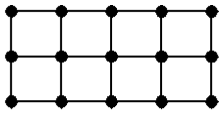
Firstly we define a classification of connected components, noted like this:



This classification also defines 12 types of connected graphs. We add an extra type noted $I.E$, corresponding to the case of isolated edges. Indeed we do not consider isolated edges as graphs of type $O_{e,o}$, since it is the only case where a player wins two points by deletion of an edge. Indices E, V and P have two possible values, noted o and e (for "odd" and "even"). There exists an additional value for the index P, noted $\emptyset$, corresponding to configurations without hanging edge (this implies there exists no edge deletion that enables to get points). We use the symbol $*$ to describe that an index can be of any value of its definition set (for instance, $E_{*,*}$ represents all connected graphs with an even number of edges). Following exemples show connected graphs of types $O_{o,o}$ and $E_{o,\emptyset}$:



$O_{o,o}$

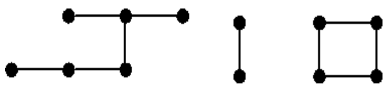


$E_{o,\varnothing}$

Now we define the general classification of game configurations, using the previous one. By playing on small exemples, it seems that winning positions are closely linked to their parity of numbers of "odd connected components" (i.e. graphs with an odd number of edges). Taking this into account, our configuration clasification is defined over three bits (p_1, p_2, p_3), where :

- the parity of the number of isolated edges is denoted by p_1
- the parity of the number of components of type $O_{o,\emptyset}$ or $O_{o,e}$ is denoted by p_2
- the parity of the number of the other "odd components", i.e. the set $\{O_{o,o}, O_{e,\emptyset}, O_{e,o}, O_{e,e}\}$, is denoted by p_3

Hence we define a 8-partition of the configurations set S. For instance, the following configuration is of type (101):



3 The evaluation function and its approximation

Now we give upper and lower bounds on $w(C)$ for any configuration C. These bounds will help us deciding whether some types of configuration are first or second-player win, and in some cases, searching for a good strategy to apply.
The configuration set can be splitted into two groups : configurations with an odd number of odd connected graphs (i.e. configurations of type (001), (010), (100) and (111)), and configurations with an even number of odd connected graphs (i.e. (000), (011), (101), (110)). This split verifies this quasi-immediate property: removing an edge from a configuration belonging to one of these two groups leads to a configuration belonging to the other group.

Making an estimate

Notation: Let C be a configuration game. $|E_{v,p}|$ defines the number of graphs of type $E_{v,p}$ composing C.
We note then $f(C)$ the following value :

$$f(C) = |O_{o,\emptyset}| + |O_{o,e}| + 2|O_{o,o}| + |O_{e,*}| + |E_{o,*}| + 2|E_{e,\emptyset}| + 2|E_{e,e}| + 2\lfloor\frac{|E_{e,o}|}{2}\rfloor$$

Besides, we note $w(p_1, p_2, p_3)$ the value of $w(C)$, if C is of type (p_1, p_2, p_3).
Before stating the main theorem, we give a useful lemma.

Lemma 1. *On a connected graph G with an odd number of edges there exists a "maximal odd chain", i.e. a chain extracted from G with an odd number of edges, where both extremities are vertices of degrees different from 2, and inner vertices of degree 2.*

Proof. We first label each edge of G with the value 1. Then we apply the following algorithm :
For each remaining vertex of degree 2, we delete it and replace both adjacent edges (labeled a and b for exemple) with a single edge labeled $a + b$, as shown below :

Finally we have a labeled graph (not necessarily simple) where each edge corresponds to a maximal chain. The label of an edge in the resulting graph equals the number of edges that it represents in the initial graph. As G had initially an odd number of edges, there exists at least one edge in the resulting graph with an odd label. This concludes the proof. □

Theorem 2. *Let C be a configuration game. The evaluation function w on C verifies following inequalities :*

$$\begin{cases} w(001) \geq f(C)+1 \\ w(010) \geq f(C) & \text{if } |E_{e,o}| \equiv 0(2) \\ \qquad\quad\ \geq f(C)+2 & \text{if } |E_{e,o}| \equiv 1(2) \\ w(100) \geq f(C)+2 \\ w(111) \geq f(C)+1 & \text{if } |E_{e,o}| \equiv 0(2) \\ \qquad\quad\ \geq f(C)+3 & \text{if } |E_{e,o}| \equiv 1(2) \end{cases} \qquad \begin{cases} w(000) \leq -f(C) \\ w(011) \leq -f(C)+1 & \text{if } |E_{e,o}| \equiv 0(2) \\ \qquad\quad\ \leq -f(C)-1 & \text{if } |E_{e,o}| \equiv 1(2) \\ w(101) \leq -f(C)+1 \\ w(110) \leq -f(C)+2 & \text{if } |E_{e,o}| \equiv 0(2) \\ \qquad\quad\ \leq -f(C) & \text{if } |E_{e,o}| \equiv 1(2) \end{cases}$$

Proof. By way of contradiction, assume that there exists a configuration C for which one of these inequalities is not true, and choose C minimal in terms of number of edges. We distinguish two cases about C, according to the group of configurations to which it belongs.

C has an even number of odd connected graphs :
As counter-exemple, C verifies an inequality of type $w(C) > -f(C) + \varepsilon$ where $\varepsilon \in \{-1, 0, 1, 2\}$. Hence, if $C \neq \emptyset$, there exists a move from C to a resulting configuration C', giving g points, such that $w(C') < f(C) - \varepsilon + g$. A contradiction will be obtained by showing that each legal move from C can not lead to such an inequality. We need then to know for each of the 13 types of connected components to which resulting graphs a move leads. The following table contains all the moves playable from a connected graph G :

Initial component	Resulting components after one move		
	Giving points	Non-disconnecting	Disconnecting
I.E.	$\emptyset$(2 pts)	-	-
$O_{o,\emptyset}$	-	$E_{o,*}$	$O_{o,*}+O_{e,*}$, $E_{o,*}+E_{e,*}$
$O_{o,e}$	$E_{e,o}$(1 pt)	$E_{o,e/o}$	$O_{o,*}+O_{e,o/e}$, $O_{o,o/e}+O_{e,\emptyset}$, $O_{o,*}+I.E.$, $E_{o,*}+E_{e,o/e}$, $E_{o,o/e}+E_{e,\emptyset}$
$O_{o,o}$	$E_{e,e/\emptyset}$(1 pt)	$E_{o,e/o}$	$O_{o,*}+O_{e,o/e}$, $O_{o,o/e}+O_{e,\emptyset}$, $O_{o,*}+I.E.$, $E_{o,*}+E_{e,o/e}$, $E_{o,o/e}+E_{e,\emptyset}$
$O_{e,\emptyset}$	-	$E_{e,*}$	$O_{o,*}+O_{o,*}$, $O_{e,*}+O_{e,*}$, $E_{o,*}+E_{o,*}$, $E_{e,*}+E_{e,*}$
$O_{e,e}$	$E_{o,o}$(1 pt)	$E_{e,o/e}$	$O_{o,*}+O_{o,o/e}$, $O_{o,o/e}+O_{o,\emptyset}$, $O_{e,*}+O_{e,o/e}$, $O_{e,o/e}+O_{e,\emptyset}$, $O_{e,*}+I.E.$, $E_{o,*}+E_{o,o/e}$, $E_{o,o/e}+E_{o,\emptyset}$, $E_{e,*}+E_{e,o/e}$, $E_{e,o/e}+E_{e,\emptyset}$
$O_{e,o}$	$E_{o,e/\emptyset}$(1 pt)	$E_{e,o/e}$	$O_{o,*}+O_{o,o/e}$, $O_{o,o/e}+O_{o,\emptyset}$, $O_{e,*}+O_{e,o/e}$, $O_{e,o/e}+O_{e,\emptyset}$, $O_{e,*}+I.E.$, $E_{o,*}+E_{o,o/e}$, $E_{o,o/e}+E_{o,\emptyset}$, $E_{e,*}+E_{e,o/e}$, $E_{e,o/e}+E_{e,\emptyset}$
$E_{o,\emptyset}$	-	$O_{o,*}$	$O_{o,*}+E_{e,*}$, $E_{i,*}+O_{e,*}$
$E_{o,e}$	$I.E.$, $O_{e,o}$(1 pt)	$O_{o,o/e}$	$O_{o,*}+E_{e,o/e}$, $O_{o,o/e}+E_{e,\emptyset}$, $E_{i,*}+O_{e,o/e}$, $E_{i,o/e}+O_{e,\emptyset}$, $E_{i,*}+I.E.$
$E_{o,o}$	$O_{e,e/\emptyset}$(1 pt)	$O_{o,o/e}$	$O_{o,*}+E_{e,o/e}$, $O_{o,o/e}+E_{e,\emptyset}$, $E_{i,*}+O_{e,o/e}$, $E_{i,o/e}+O_{e,\emptyset}$, $E_{i,*}+I.E.$
$E_{e,\emptyset}$	-	$I.E.$, $O_{e,*}$	$O_{o,*}+E_{o,*}$, $O_{e,*}+E_{e,*}$
$E_{e,e}$	$O_{o,o}$(1 pt)	$O_{e,o/e}$	$O_{o,*}+E_{o,o/e}$, $O_{o,o/e}+E_{o,\emptyset}$, $O_{e,*}+E_{e,o/e}$, $O_{e,o/e}+E_{e,\emptyset}$, $E_{e,*}+I.E.$
$E_{e,o}$	$O_{o,e/\emptyset}$(1 pt)	$O_{e,o/e}$	$O_{o,*}+E_{o,o/e}$, $O_{o,o/e}+E_{o,\emptyset}$, $O_{e,*}+E_{e,o/e}$, $O_{e,o/e}+E_{e,\emptyset}$, $E_{e,*}+I.E.$

Suppose now C of type (000) with $|E_{e,o}| \equiv 0(2)$, and such as $w(C) > -f(C)$. If there exists no move from C, then C does not contain any edge, and thus $w(C) = f(C) = 0$, which yields a contradiction. Otherwise, there exists a move from C to a resulting configuration C' and giving g points, such as $w(C') < f(C) + g$ (*). We will consider C each move appearing in the table above and show that (*) is never verified.
As an exemple, consider the removing of an edge from a $O_{o,e}$ component belonging to C. There are several possible resulting sets :

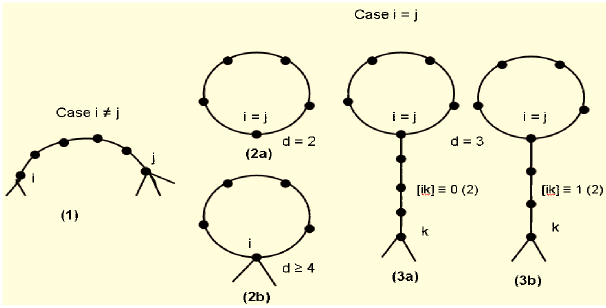
- Deleting a hanging edge (winning one point), lets a $E_{e,o}$ component to the other player. The resulting configuration is of type (010) with $|E_{e,o}| \equiv 1(2)$. As C' has less edges than C and by minimality of C, we have $w(C') \geq f(C') + 2$. Besides we have $f(C') = f(C) - 1$, hence we conclude to a contradiction with (*).
- Playing from $O_{o,e}$ to $E_{o,o}$ (with a gain $g = 0$). The resulting configuration is a (010), with $|E_{e,o}| \equiv 0(2)$. Then we have by minimality of C : $w(C') \geq f(C') = f(C)$, confirming the contradiction.
- Removing a disconnecting edge from a $O_{o,e}$ can lead to $O_{o,\emptyset} + O_{e,e}$ for exemple ($g = 0$). The (001) resulting configuration verifies $w(C') \geq f(C') + 1 = f(C) + 2$, and ensures the contradiction with (*).
- For each other move from $O_{o,e}$ and from each of the 12 other types, we show similarly that we get a contradiction with (*).

This implies that C is not of type (000) with $|E_{e,o}| \equiv 0(2)$. Then we suppose that C is of type of each other configuration of this group (i.e. (011), (101), and (110), with $|E_{e,o}|$ parity equal to 0 or 1). By the same way as explained above, a carefully exploration of all the moves listed in the table (that we do not detail in this paper) proves that inequalities of type $w(C) > -f(C) + \varepsilon$ are contradicted, so that the minimal counter-exemple C does not belong to this group of configurations.

C has an odd number of odd connected graphs :
The second part of the proof is closely linked to the strategy to apply when the configuration contains an odd number of "odd connected graphs". In this case, the counter-exemple C verifies an inequality of type $w(C) < f(C) + \varepsilon$, $\varepsilon \in \{0, 1, 2, 3\}$. This means that for every move from C to a resulting configuration C' and giving g points, we have $w(C') > -f(C) - \varepsilon + g$. We will yield a contradiction by finding a move for which this inequality is not true. Consider all different cases about C :

- C is of type (100) and verifies $w(C) < f(C) + 2$. Then for every move from C to C' with a gain g, we have $w(C') > -f(C) - 2 + g$. As $p_1 = 1$ in this configuration type, we choose the deletion of an isolated edge, giving two points. The resulting set C' is of type (000) and verifies $w(C') \leq -f(C') = -f(C)$ as C is minimal, which is in contradiction with the inequality above.
- C is of type (111). We choose the same type of move, i.e. the deletion of an isolated edge. No matter the parity of $|E_{e,o}|$, we get a contradiction by the same way.
- C is of type (010). The parity of $|E_{e,o}|$ in C defines two cases :
  - $|E_{e,o}| \equiv 1(2)$. C verifies $w(C) < f(C) + 2$ and for every move from C to C' (with a gain g), we have $w(C') > -f(C) - 2 + g$. We can also remove a hanging edge from a $E_{e,o}$ component belonging to C, giving one point, and creating a $O_{o,\emptyset/e}$. The resulting configuration C' is of (000) type, and verifies $w(C') \leq -f(C') = -f(C) - 1$, which yields a contradiction.
  - $|E_{e,o}| \equiv 0(2)$. We have $w(C) < f(C)$, implying for every move from C to C' giving g points: $w(C') > -f(C) + g$. In a (010) configuration, there exists a $O_{o,\emptyset}$ or a $O_{o,e}$ component. If there exists a $O_{o,\emptyset}$ component, we remove a non-disconnecting edge on it. Such an edge exists : if all edges were disconnecting ones, the component would be a tree, contradicting $p = \emptyset$ in a $O_{o,\emptyset}$. This move gives no point and lets a resulting component $E_{o,*}$ in a (000) configuration C', verifying $w(C') \leq -f(C') = -f(C)$. Otherwise, there exists a $O_{o,e}$ component, and we can remove a hanging edge to give a $E_{e,o}$ in a (000) configuration, taking one point. The resulting configuration C' verifies $w(C') \leq -f(C') = -f(C) + 1$. In both cases, the contradiction is guaranteed.
- C is of type (001). If $|E_{e,o}| \equiv 1(2)$, we choose the same move as previously to get the contradiction, i.e. the deletion of a hanging edge from a $E_{e,o}$ component. If $|E_{e,o}| \equiv 0(2)$, we only know that C contains a component of type $O_{o,o}$, $O_{e,e}$, $O_{e,o}$ or $O_{e,\emptyset}$. Moreover, $w(C) < f(C) + 1$, and for every move from C to C' giving g points, we have $w(C') > -f(C) - 1 + g$. Two cases are considered :
  - There exists a component with at least a hanging edge (i.e. $O_{o,o}$, $O_{e,e}$ or $O_{e,o}$). As previously chosen, the deletion of a hanging edge gives a (000) resulting set with one point, and leads to a contradiction.
  - C does not contain such a component: we remove an edge belonging to a $O_{e,\emptyset}$. Almost each move from a $O_{e,\emptyset}$ component (appearing in the table) to a resulting configuration C' leads to a inequality of the form $w(C') \leq -f(C) + 1$, which is in contradiction with the initial inequality of this case. The only case without contradiction is the move that creates a single $E_{e,o}$. We have thus to prove that it is always possible to find a move different

from this one.
First of all, we use lemma 1, which states that there exists a maximal odd chain taken from the $O_{e,\emptyset}$ component. This chain could eventually be a cycle. We note i and j both extremities of it (it may be that $i = j$). See below different possibilities about this chain :



* First case is $i \neq j$. As a $O_{e,\emptyset}$ component does not contain vertices with degree one, i and j are vertices of degrees at least 3. As every inner vertex is of degree 2, if we remove an edge from the chain, one or two hanging trees will appear, whose total number of edges is even. We can not create a single $E_{e,o}$.
* Consider the case where the chain is a cycle and note i a vertex of this cycle of maximum degree. Figures ($2a$) and ($2b$) present cycles where respectively $deg(i) = 2$ and $deg(i) \geq 4$. In both cases, we remove an edge of this cycle, giving a resulting graph where hanging trees created have an even total number of edges. If $deg(i) = 3$, we consider the length l of the chain between i and k (where k is the first vertex of this chain with a degree upper than 2). If l is even (case ($3a$)), we can play on the cycle, as the resulting component contains a hanging tree with an even number of edges. Otherwise, as we can not play on the cycle (or a single $E_{e,o}$ would appear), we remove an edge from the chain between i and k (a such edge exists as the chain has an odd length). The graph is disconnected, so that a single $E_{e,o}$ can not appear.

We have a contradiction for each case, ending the proof. □

This result leads to the following corollary, about winning positions of this game :

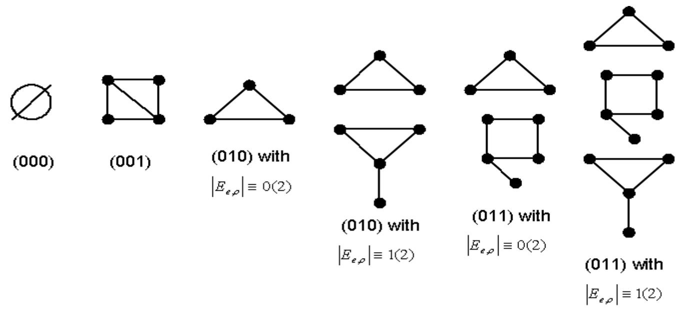
Corollary 3. *Configurations with an odd number of "odd connected components" are winning for the first player. Configurations of type* (000) *and* (101) *are non-winning. Those of type* (011) *and* (110) *when* $|E_{e,o}| \equiv 1(2)$ *are losing ones.*

Proof. As $f(C) \geq 0$, it seems clear that configurations with an odd number of "odd connected components" have a lower bound of their evaluation function upper than 0 (notice that $f(010) \geq 1$).
If a configuration C is of type (101), then $f(C) \leq 1$, so that we can assert that $w(000)$ and $w(101)$ are lower or equals to zero, which means that they are not winning.
Concerning the other configurations, we have $f(011) \geq 2$ and $f(110) \geq 1$, leading to the conclusion.

□

Notice that configurations of type (110) with an even number of $E_{e,o}$ components are the only ones of their group that are sometimes winning.

According to this choice of parameters, the bounds given by the theorem are the best ones that we can get. For each configuration type, we have found an exemple whose value equals the announced bound. Such exemples are following ones :



Exemples of configurations of type (1..) whose value equals the bound are obtained by adding an isolated edge to each corresponding exemple of type (0..) presented above.

For each winning configuration, we can not decide what the best move is, but we know that we can win at least the number of points announced by the lower bound of the evaluation function, which is enough to win the game.
On the other hand, we have no suggested strategy to apply when the configuration has an even number of "odd connected components". In most of these cases, it would consist in finding the way to lose "with honors". For exemple, it seems reasonnable to remove an isolated edge when the configuration is of type (101) or (110). What about both other types ?
Finding lower bounds for this group of configurations would help us answering to this question. However, the configuration coding choice over 3 bits is not good enough to get such bounds. Actually this problem can be expressed as a linear system, from which we extracted a small sub-system with no solution. A larger coding system (over 13 bits at the most) may lead us finding both lower and upper bounds of the evaluation function, but would give a too big number of configuration types, which makes the proof not very exciting. Besides, another connected graphs coding could be envisaged to improve these results.

We precise that "Le Pic'arete" can be played on line at the following web address:
http://www-leibniz.imag.fr/LAVALISE/PicArete/index.html

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