Labeled packings of graphs ⋆

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Abstract

Graph packing generally considers unlabeled graphs. In this paper, we introduce a new variant of the graph packing problem, called the *labeled packing of graphs*. In particular, we present some results and conjectures about cycles.

Key words:
Packing of graphs, Labeled packing, Permutation.

1 Introduction and definitions

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. For a graph $G = (V, E)$, we use $V(G)$, $E(G)$ to denote its vertex and edge sets, respectively. Given $V' \subseteq V$, the subgraph $G[V'] = (V', E')$ denotes the subgraph of G induced by V', *i.e.,* E' contains all the edges of E which have both end vertices in V'. If a graph G has order n and size m, we say that G is an (n, m)-graph. An independent set of G is a subset of nodes $X \subseteq V$, such that no two nodes in X are adjacent. An independent set is maximal if no node can be added without violating independence. An independent set of maximum cardinality is called maximum independent set. For undefined terms, we refer the reader to [2].

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A *permutation* σ is a one-to-one mapping of a set S into itself. We say that an element e of S is a *fixed point* of a permutation σ if $\sigma(e) = e$. Let $G_1, G_2, \ldots, G_k$ be k graphs of order n. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist permutations $\sigma_i : V(G_i) \longrightarrow V(K_n)$, $i = 1, \ldots, k$, such that $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the mapping $\sigma_i^* : E(G_i) \longrightarrow E(K_n)$ is the one induced by σ_i. A packing of k copies of a graph G will be called a *k-placement* of G. A packing of two copies of G (*i.e.*, a 2-placement) is called an *embedding* of G (into its complement $\overline{G}$). In other words, an embedding of a graph G is a permutation σ on $V(G)$ such that if an edge vu belongs to $E(G)$ then $\sigma(v)\sigma(u)$ does not belong to $E(G)$. If there exists an embedding of G we say that G is *embeddable.*

Graph packing is a well known field of graph theory that has been considerably investigated in the literature. In 1978, Bollobás and Eldridge [1] made the following conjecture about the existence of a k-packing for graphs of small size, which is widely considered to be one of the most important open problems in graph packing theory.

Conjecture 1 (Bollobás and Eldridge [1]) *Let $G_1, \ldots, G_k$ be k graphs of order n. If $|E(G_i)| \leq n - k$, $i = 1, 2 \ldots k$, then $G_1, \ldots, G_k$ are packable into K_n.*

The cases $k = 2$ and $k = 3$ of this conjecture were proved in [13] and [11], respectively (the case $k = 2$ was proved also in [1], independently). That only two cases are solved shows the hardness of this problem. The following theorem gives a general result about embedding of $(n, n-1)$-graphs.

Theorem 2 (Burns and Schuster [3]) *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 1$ then either G is embeddable or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.*

A similar result about embeddings of (n, n)-graphs is proved in [5]. Recently, Żak [18] studied the packing problem for all k and proved the following result.

Theorem 3 (Żak [18]) *Let k be a positive integer and G be a graph of order $n \geq 2(k-1)^3$. If $|E(G)| \leq n - 2(k-1)^3$, then G is k-placeable.*

The graph packing theory also investigates two important directions concerning graphs with bounded maximum degree and bounded girth. The two main conjectures in this area were posed by Bollobás and Eldridge [1] (and independently by Catlin [4]) and Faudree et al. [5], respectively.

Conjecture 4 (Bollobás and Eldridge [1], Catlin [4]) *Let G_1 and G_2 be two graphs of order n with maximum degrees $\triangle_1$ and $\triangle_2$. If $(\triangle_1+1)(\triangle_2+1) \leq n + 1$, then there is a packing of G_1 and G_2.*

Conjecture 5 (Faudree, Rousseau, Schelp and Schuster [5]) *If a graph G is a non-star graph without cycles of length $m \leq 4$, then G is embeddable.*

Some significant progress related to Conjecture 4 and Conjecture 5 has recently been obtained in [6,8,9].

When considering the placement of specific families of graphs such as trees, important results have been found for the placement of two and three copies of a given tree (see [10,12,14]). An overview of graph packing can be found in the survey papers of Woźniak [16] and Yap [17]. However, the majority of existing works focuses on unlabeled graphs. In this paper, we study the packing problem for a vertex labeled graph. To the best of our knowledge, this variant has not been studied so far. Here is the definition of this new packing problem.

Definition 1 *Let p be an integer greater than one and let $G_1, G_2, \ldots, G_k$ be k copies of a graph $G = (V, E)$. Let f be a mapping from $V(G)$ to the set of labels $\{1, 2, \ldots, p\}$. The mapping f is called a p-labeled packing of k copies of G into K_n if there exist permutations $\sigma_i : V(G_i) \longrightarrow V(K_n)$, where $i = 1, \ldots, k$, such that:*

(1) $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for all $i \neq j$.
(2) for every vertex v of G, we have $f(v)=f(\sigma_1(v))=f(\sigma_2(v)) \cdots =f(\sigma_k(v))$.

For example, the following figure presents a 5-labeled packing of two copies of $C_4 \cup P_3$, where the dotted edges represent the second copie of $C_3 \cup P_3$.

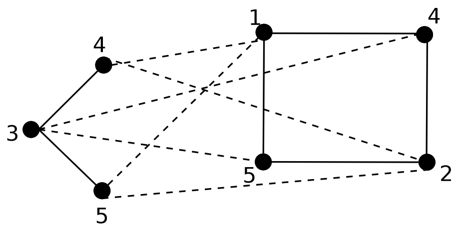


Figure 1. A 5-labeled packing of two copies of $C_4 \cup P_3$.

From the previous definition, we define our packing parameter as follows: the maximum positive integer p for which G admits a *p-labeled packing* of k copies of G is called the *k-labeled packing number* of G and is denoted by $\lambda^k(G)$.

Naturally, the existence of a packing of k copies of a graph G is a necessary condition for the existence of a *p-labeled packing* of k copies of G (where $p \geq 1$). Indeed, it suffices to choose the mapping $f : V(G) \longrightarrow \{1\}$. Throughout this paper, a labeled packing of two copies of G will be called a labeled embedding of G.

Let G be a graph of order n and f be a mapping from $V(G)$ to $\{1, 2, \ldots, p\}$.

We define $S(f)$ as the subset of labels that occur only once in $f(V(G))$ and let $V_{S(f)}$ consist of the vertices colored with the labels of $S(f)$. Thus it is obvious that $|V_{S(f)}| = |S(f)|$.

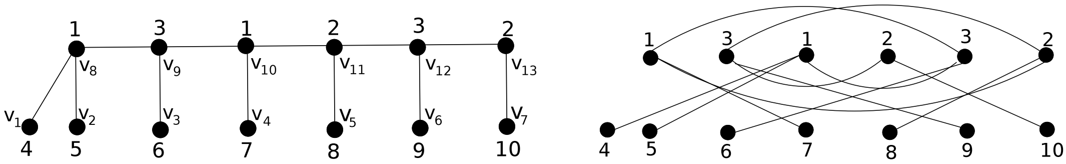
The following lemma gives an upper bound on the labeled embedding number $\lambda^2(G)$.

Lemma 6 *Let G be a graph of order n and let I be a maximum independent set of G. If there exists an embedding of G, then*

$$\lambda^2(G) \leq |I| + \lfloor \frac{n - |I|}{2} \rfloor$$

Proof. Let f be a mapping from $V(G)$ to $\{1, 2 \ldots, p\}$ corresponding to a *p-labeled embedding* of G with p maximum, (*i.e.*, $p = \lambda^2(G)$). A necessary condition for the existence of a *p-labeled embedding* of G is that for every two adjacent vertices of G, one of their labels must occur at least twice in $f(V(G))$. Hence, it is easy to see that all vertices of $V_{S(f)}$ form an independent set in G. Then, we have $|V_{S(f)}| \leq |I|$ and $p \leq |V_{S(f)}| + \lfloor \frac{n - |V_{S(f)}|}{2} \rfloor$. Since $g(x) = x + \lfloor \frac{n-x}{2} \rfloor$ is an increasing function of x, it follows that $p \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor$, giving the desired result. $\square$

It is clear that any labeled embedding construction (the labeling function f and the permutation σ) that achieves the upper bound of Lemma 6 must consider the vertices of $V_{S(f)}$ as fixed-points under the permutation σ, which then requires to define a $\lfloor \frac{|V(G) \setminus V_{S(f)}|}{2} \rfloor$-labeled embedding without fixed points for the subgraph induced by the set of vertices $V(G) \setminus V_{S(f)}$. For example, let us consider the caterpillar T of Figure 2(a). From Lemma 6, we have $\lambda^2(T) \leq 10$. This upper bound can be reached by finding a 3-labeled embedding without fixed points for the central path of T (Figure 2(b)).



(a) (b)

Figure 2. (a) A caterpillar T, (b) A 10-labeled embedding of T

Any permutation σ of a finite set can be written as the disjoint union of cycles (two cycles are disjoint if they do not have any common element). Here, a cycle $(a_1, a_2, \ldots, a_n)$ is a permutation sending a_i to a_{i+1} for $1 \leq i \leq n-1$ and a_n to a_1. This representation is called the *cyclic decomposition of σ* and is denoted by $C(\sigma)$. In this context, the cycles of length one correspond to fixed points of σ. For example, the cyclic decomposition of the permutation induced by the labeled embedding of T (in Figure 2) is: $\{(v_1), (v_2), (v_3), (v_4), (v_5), (v_6), (v_7), (v_8, v_{10}), (v_9, v_{12}), (v_{11}, v_{13})\}$.

We now explain a fundamental congruence relation between the labeled embedding number and the permutation structure. For any labeled embedding of a graph G induced by a permutation σ, we can easily see that the vertices of every cycle of $C(\sigma)$ share the same label. Hence, we can say that the labeled embedding number of G denotes the maximum number of cycles induced by a permutation of G.

As we have seen previously, the packing graph is very hard problem; only some particular results are known for graphs of small size ($(n, n-q)$ graphs where $q \geq 0$). Two studies are possible for the labeled graph packing problem: (i) packing important number of sample graphs (or copies of a given graph) in a complete graph; or (ii) packing a fixed number (usually two) of non trivial graph in a complete graph. For the first study of labeled packing of graphs, we opt for the first possibility. We will give some results concerning the labeled packing number of k copies of cycles (where $k \geq 2$).

Given a graph G, the generalization of Lemma 6 for any k is not natural. Yet, it becomes easier when G is a cycle. We know that any complete graph K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles if n is odd and $\frac{n-2}{2}$ Hamilton cycles plus a perfect matching if n is even (see [7]). Thus, C_n cannot admit a packing of k copies in K_n if $n \leq 2k$.

Lemma 7 *For every cycle C_n of order $n \geq 2k+1$, we have*

$$\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$$

Proof. Let f be a mapping from $V(G)$ to $\{1, 2, \ldots, p\}$ corresponding to a p-labeled packing of k copies of C_n with p maximum. Let us introduce the following notation: for a subset S of vertices of G, we denote by $N(S)$ the set of vertices of $G \setminus S$ adjacent to at least one vertex of S. Then, two necessary conditions for the existence of *p-labeled packing* of k copies of C_n are that:

(1) the labels of $N(V_{S(f)})$ must occur at least k times in $f(V(C_n))$.
(2) the labels of $V \setminus (V_{S(f)} \cup N(V_{S(f)}))$ must occur at least twice in $f(V(C_n))$.

Hence, we have

$$p = |V_{S(f)}| + \lfloor \frac{|N(V_{S(f)})|}{k} \rfloor + \lfloor \frac{n - |V_{S(f)}| - |N(V_{S(f)})|}{2} \rfloor$$

We can see that,

$$|V_{S(f)}| \leq |N(V_{S(f)})| \leq 2|V_{S(f)}|$$

We consider now that $|N(V_{S(f)})| = |V_{S(f)}| + x$, where $0 \leq x \leq |V_{S(f)}|$. Then, we have

$$p = \lfloor \frac{|V_{S(f)}| + x}{k} \rfloor + \lfloor \frac{n - x}{2} \rfloor$$

Since $k \geq 2$, the maximum value of p is obtained when $|V_{S(f)}|$ is at a maximum and x is at a minimum, *i.e.*, $|V_{S(f)}| = \lfloor \frac{n}{2} \rfloor$ and $x = 0$. Hence, we obtain that $p = \frac{n}{2} + \lfloor \frac{n}{2k} \rfloor$, giving the desired result. $\square$

The rest of this paper is organized in three sections as follows: in Section 2, we show the exact value of $\lambda^k(C_n)$ for $n \geq 4k$. Section 3 presents a conjecture for the cycles of order at most $4k - 1$ and we will prove this conjecture for some particular cases. Finally, we conclude the paper by summarizing our results.

2 Labeled-packing of cycles of order at least $4k$

In this section, we show the exact value of the labeled packing number of k copies of cycles of order $n \geq 4k$. It is done by proving that the upper bound of Lemma 7 is reached for all $n \neq 4k + x$, where $k > 2$ and $x = 2 \pmod 4$.

Theorem 8 *For every cycle C_n of order $n = 2km + x$, where $k, m \geq 2$ and $x < 2k$, we have*

$$\lambda^k(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } (x \bmod 4, m) = (2, 2) \text{ and } k > 2. \\ \lfloor \frac{n}{2} \rfloor + m + 1 & \text{if } x = 2k - 1. \\ \lfloor \frac{n}{2} \rfloor + m & \text{otherwise.} \end{cases}$$

Proof. Several cases are considered in this proof according to the values of x and m. In the following figure, we outline the general scheme of our proof.

Case 1 : $x \neq 2 \pmod 4$. Let $C_n = (v_1, v_2, \ldots, v_n)$ be a cycle of order n, where $n \geq 4k$. From Lemma 7, we have $\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$, it then suffices to prove that C_n admits a $(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor)$-labeled packing of k copies. In order to simplify our proof, we consider three subcases according to the values of m and x as follows:

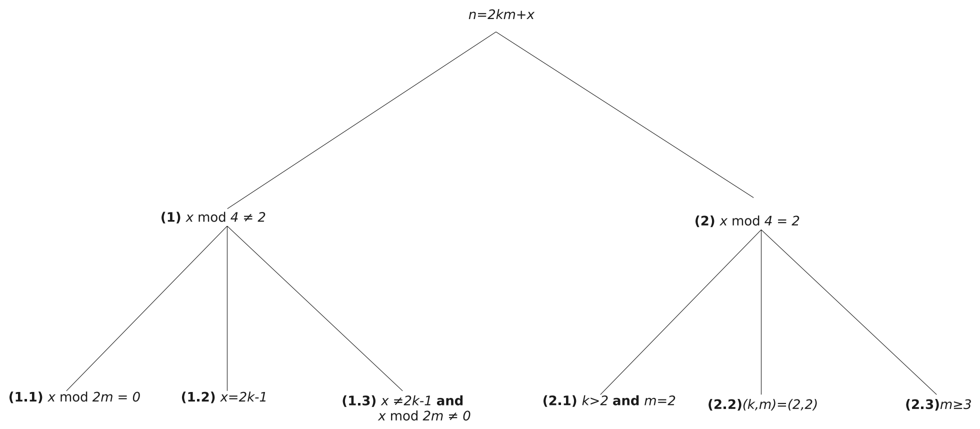


Figure 3. Proof structure.

Subcase 1.1: $x = 0 \pmod{2m}$. Let $B = \{b_0, b_1, \ldots, b_{p-1}\}$ be a partition of $V(C_n)$ into p sets, where all sets b_i of B have the same cardinalty $2m$. Then, we have $p = k + \frac{x}{2m}$ such that for $0 \le i \le p-1$, $b_i = \{v_1^i, v_2^i, \ldots, v_{2m}^i\}$, where $v_j^i = v_{2im+j}$. These notations are illustrated in the example of Figure 4, where $n = 16$ and $k = 3$.

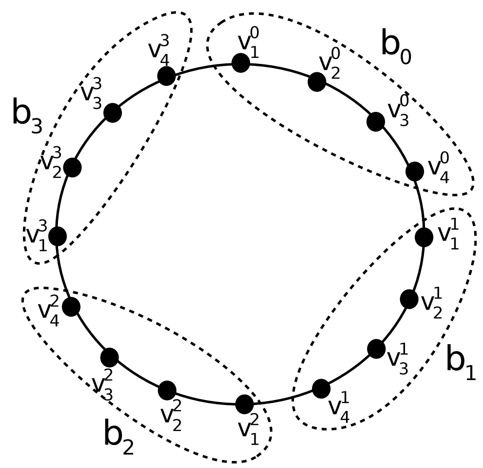


Figure 4. Partition of $V(C_{16})$ for $k = 3$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m\}$ is now defined as follows: for every set b_i in B, let

$$f(v_{2j+1}^i) = im + j + 1 \quad \text{for } 0 \le j \le m-1.$$
$$f(v_{2j}^i) = \frac{n}{2} + j \quad \text{for } 1 \le j \le m.$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n)$, $\sigma^2(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where for $0 \le t \le$

$k-1$,

$$\sigma^t(v^i_{2j+1}) = v^i_{2j+1} \qquad \text{for all } (i,j) \text{ satisfying } 0 \le i \le p-1 \text{ and } 0 \le j \le m-1.$$
$$\sigma^t(v^i_{2j}) = v^{(i+t) \bmod p}_{2j} \qquad \text{for all } (i,j) \text{ satisfying } 0 \le i \le p-1 \text{ and } 1 \le j \le m.$$

Figure 5 presents a 10-labeled-packing of three copies of C_{16}.

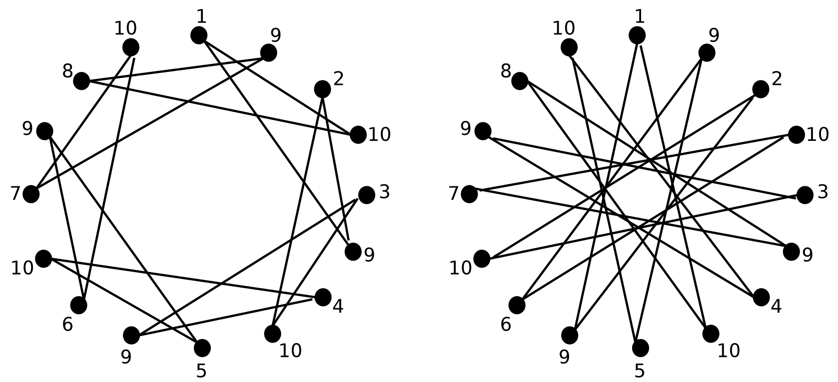


Figure 5. A 10-labeled-packing of three copies of C_{16}: (a) $\sigma^1(C_{16})$, (b) $\sigma^2(C_{16})$.

From the set of permutations, we can easily see that for every vertex v_i in $V(C_n)$, we have $f(v_i) = f(\sigma^1(v_i)) = f(\sigma^2(v_i)) = \cdots = f(\sigma^{k-1}(v_i))$. It then remains to show that the set of cycles $\sigma^0(C_n) = C_n, \sigma^1(C_n), \sigma^2(C_n) \ldots, \sigma^{k-1}(C_n)$ are edge-disjoint into K_n. To prove this, it suffices to show that for every vertex v_i of $V(C_n)$, we have:

For $0 \le t \ne l \le k-1$, if $\sigma^t(v_i) = \sigma^l(v_i)$, then $\sigma^t(v_{i+1})$ and $\sigma^t(v_{i-1})$ must be different from both $\sigma^l(v_{i+1})$ and $\sigma^l(v_{i-1})$.

From the set of permutations, we can observe that for every two distinct integers $l, t \in \{0, 1, \ldots, k-1\}$ and for every set b_i of B, we have $\sigma^l(v^i_{2j+1}) = \sigma^t(v^i_{2j+1}) = v^i_{2j+1}$, where $0 \le j \le m-1$. Then for every set b_i of B, we consider two cases according to whether $j \ne 1$ or $j = 1$.

(1) for $j = 3, 5 \ldots 2m-1$, we have $\sigma^l(v^i_{j+1}), \sigma^l(v^i_{j-1}) \in b^{(l+i) \bmod p}$ and $\sigma^t(v^i_{j+1})$, $\sigma^t(v^i_{j-1}) \in b^{(t+i) \bmod p}$. It follows that $(l+i) \bmod p \ne (t+i) \bmod p \ne i$.
(2) for $j = 1$, we can easily see that $\sigma^l(v^i_2)$ and $\sigma^l(v^{i-1}_{2m})$ are both different from $\sigma^t(v^i_2)$ and $\sigma^t(v^{i-1}_{2m})$.

Hence, according to the two previous observations, we can see that C_n, $\sigma^1(C_n)$, $\sigma^2(C_n)$, $\ldots$, $\sigma^{k-1}(C_n)$ are k pairwise edge-disjoint cycles, thus the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n.

Subcase 1.2: $x = 2k-1$. In this case, the partition $B = \{b_0, b_1, \ldots, b_{k-1}\}$ of $V(C_n)$ is defined as follows: for $0 \le i \le k-2$, $b_i = \{v^i_1, v^i_2, \ldots, v^i_{2(m+1)}\}$ and

$b_{k-1} = \{v_1^{k-1}, v_2^{k-1}, \ldots, v_{2m+1}^{k-1}\}$, where $v_j^i = v_{2i(m+1)+j}$.

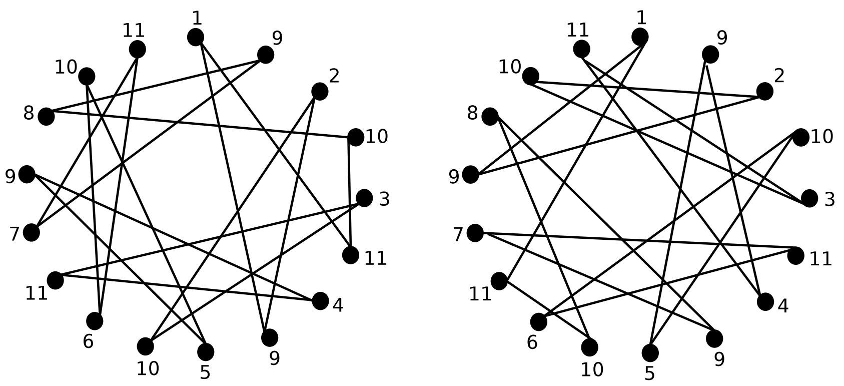
A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m + 1\}$ is now defined as follows: for every set b_i in B, let

$$
\begin{aligned}
f(v_{2j+1}^i) &= i(m+1) + j + 1 && \text{for } 0 \le j \le m \text{ and } (i,j) \ne (k-1, m). \\
f(v_{2m+1}^{k-1}) &= \tfrac{n-1}{2} + m + 1. \\
f(v_{2j}^i) &= \tfrac{n-1}{2} + j && \text{for } 1 \le j \le m+1 \text{ and } (i,j) \ne (k-1, m+1).
\end{aligned}
$$

Let $\sigma^0(C_n),\ \sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le k-1$, let

$$
\begin{aligned}
\sigma^t(v_{2j+1}^i) &= v_{2j+1}^i && \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-1, 0 \le j \le m \\
& && \text{and } (i,j) \ne (k-1, m). \\
\sigma^t(v_{2j}^i) &= v_{2j}^{(i+t) \bmod k} && \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-1, 1 \le j \le m+1 \\
& && \text{and } (i,j) \ne (k-t-1, m+1). \\
\sigma^t(v_{2m+1}^{k-1}) &= v_{2m+2}^{t-1}. \\
\sigma^t(v_{2m+2}^{k-t-1}) &= v_{2m+1}^{k-1} = v_n.
\end{aligned}
$$

Figure 6 shows a 11-labeled-packing of three copies of C_{17}.



(a) (b)

Figure 6. A 11-labeled-packing of three copies of C_{17}: (a) $\sigma^1(C_{17})$, (b) $\sigma^2(C_{17})$.

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m + 1)$-labeled-packing of k copies of C_n.

Subcase 1.3: $x = 0$ (mod) $2m$ and $x \ne 2k-1$. Let $B = \{b_0, b_1, \ldots, b_{p-1}\}$ be a partition of $V(C_n)$ into $p = k + \lceil \frac{x}{2m} \rceil$ sets such that for $0 \le i \le p-2$, $b_i = \{v_1^i, v_2^i, \ldots, v_{2m}^i\}$ and $b_{p-1} = \{v_1^{p-1}, v_2^{p-1}, \ldots, v_{x \bmod 2m}^{p-1}\}$, where $v_j^i = v_{2im+j}$. We let $r = x \bmod 2m$.

A mapping f from $V(C_n)$ to $\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor + m\}$ is defined as follows:

$f(v^i_{2j+1}) = im + j + 1$ for all (i, j) satisfying $0 \le i \le p-2, 0 \le j \le m-1$.

$f(v^i_{2j}) = \lfloor \frac{n}{2} \rfloor + j$ for all (i, j) satisfying $0 \le i \le p-2, 1 \le j \le m$.

$f(v^{p-1}_{2j+1}) = (p-1)m + j + 1$ for $0 \le j \le \lceil \frac{r-1}{2} \rceil - 1$.

$f(v^{p-1}_{2j}) = \lfloor \frac{n}{2} \rfloor + j$ for $1 \le j \le \lfloor \frac{r-1}{2} \rfloor$.

$f(v^{p-1}_{r}) = \lfloor \frac{n}{2} \rfloor + m$.

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \dots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \dots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le k-1$, let

$\sigma^t(v^i_{2j+1}) = v^i_{2j+1}$ for all (i, j), $0 \le i \le p-2$ and $0 \le j \le m-1$.

$\sigma^t(v^i_{2j}) = v^{(i+t) \bmod p}_{2j}$ for all (i, j), $0 \le i \le p-2$ and $1 \le j \le \lfloor \frac{r-1}{2} \rfloor$.

$\sigma^t(v^i_{2j}) = v^{(i+t) \bmod (p-1)}_{2j}$ for all (i, j), $0 \le i \le p-2$ and $\lfloor \frac{r-1}{2} \rfloor + 1 \le j \le m-1$.

$\sigma^t(v^i_{2m}) = v^{(i+t) \bmod p}_{2m}$ for $0 \le i \ne p-t-1 \le p-2$.

$\sigma^t(v^{p-t-1}_{2m}) = v^{p-1}_{r} = v_n$.

$\sigma^t(v^{p-1}_{2j+1}) = v^{p-1}_{2j+1}$ for $0 \le j \le \lceil \frac{r-1}{2} \rceil - 1$.

$\sigma^t(v^{p-1}_{2j}) = v^{t-1}_{2j}$ for $1 \le j \le \lfloor \frac{r-1}{2} \rfloor$.

$\sigma^t(v^{p-1}_{r}) = v^{t-1}_{2m}$.

Figure 7 shows a 9-labeled-packing of three copies of C_{15}.

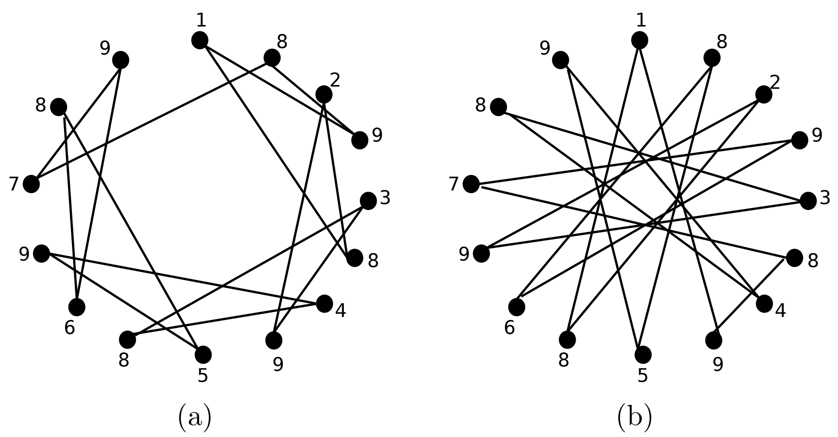


Figure 7. A 9-labeled-packing of three copies of C_{15}: (a) $\sigma^1(C_{15})$, (b) $\sigma^2(C_{15})$.

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n.

Case 2: $x = 2 \pmod 4$. We consider three subcases according to the value of k and m as follows:
Subcase 2.1: $k > 2$ **and** $m = 2$. We first prove that $\lambda^k(C_{4k+x}) < \frac{n}{2}+2$. From Lemma 7, it follows that the value $\frac{n}{2}+2$ is an upper bound of $\lambda^k(C_{4k+x})$. We assume that there exists a mapping $f : V(C_{4k+x}) \longrightarrow L = \{1, 2, \ldots, \frac{n}{2}+2\}$ which is a $(\frac{n}{2}+2)$-labeled-packing of C_{4k+x}. Let $\sigma^0(C_{4k+x}), \sigma^1(C_{4k+x}), \ldots, \sigma^{k-1}(C_{4k+x})$ be a packing of k copies of C_{4k+x} into K_{4k+x} under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively. We know that the vertices colored with the labels of S_f form an independent set in C_{4k+x} (implying that $|S_f| \leq \frac{n}{2}$). We consider two cases according to the cardinality of S_f as follows:

(a): $|S_f| < \frac{n}{2}$. From Lemma 7, $2k + \frac{x}{2} + 2$ is an upper bound for $\lambda^k(C_{4k+x})$. Using the same neccesary conditions defined in the proof of Theorem 7, we can derive the following equation (we recall that $N(V_{S(f)})$ is the set of vertices of $C_{4k+x} \setminus V_{S(f)}$ adjacent to at least one vertex of $V_{S(f)}$).

$$2k+\frac{x}{2}+2 = max\{\lfloor\frac{|V_{S(f)}|+y}{k}\rfloor+\lfloor\frac{n-y}{2}\rfloor, 0 \leq |V_{S(f)}| \leq \lfloor\frac{n}{2}\rfloor-1, 1 \leq y \leq |V_{S(f)}|\}$$

Since $k \geq 2$, the maximum of the second term of the inequality is obtained when $|V_{S(f)}|$ is at a maximum and y is at a minimum, *i.e.*, $|V_{S(f)}| = \lfloor\frac{n}{2}\rfloor - 1$ and $y = 1$. Hence, we obtain that

$$2k + \frac{x}{2} + 2 = \lfloor\frac{\lfloor\frac{n}{2}\rfloor}{k}\rfloor + \lfloor\frac{n-1}{2}\rfloor$$

From hypothesis, we have $n = 4k + x$ is even and $x < 2k$, then

$$2k + \frac{x}{2} + 2 \leq 2 + \lfloor\frac{n}{2}\rfloor - 1 = 2k + \frac{x}{2} + 1$$

Hence, this leads to a contradiction for $|S_f| < \frac{n}{2}$.

(b): $|S_f| = \frac{n}{2}$. It means that the vertices colored with the labels of S_f form a maximum independent set in C_{4k+x}. Then the two labels of $L \setminus S_f$ are attributed to the remaining vertices such that each label occurs at least k times. With this labeling scheme, there exist necessarily three vertices v_{i-1}, v_i and v_{i+1} such that $f(v_i) \in S_f$ and $f(v_{i-1}), f(v_{i+1}) = c \in L \setminus S_f$. This requires that the label c must occur at least $2k$ times in C_{4k+x}, yielding a contradiction. Hence $\lambda^k(C_{4k+x}) < \frac{n}{2} + 2$.

We now prove that $\lambda^k(C_{4k+x}) = \frac{n}{2} + 1$. Let f be the mapping from $V(C_n)$ to

$\{1, 2, \ldots, \frac{n}{2}+1\}$ defined as follows:

$$f(v_{2j+1}) = j+1 \quad \text{for } j = 0, 1, \ldots, \tfrac{n}{2}-1.$$
$$f(v_{2j}) = \tfrac{n}{2}+1 \quad \text{for } j = 1, 2 \ldots, \tfrac{n}{2}.$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le k-1$:

$$\sigma^t(v_{2j+1}) = v_{2j+1} \quad \text{for } j = 0, 1, \ldots, \tfrac{n}{2}-1.$$
$$\sigma^t(v_{2j}) = v_{2(j+2t) \bmod n} \quad \text{for } j = 1, 2 \ldots, \tfrac{n}{2}.$$

According to this scheme, Figure 8 presents a 8-labeled-packing of three copies of C_{14}.

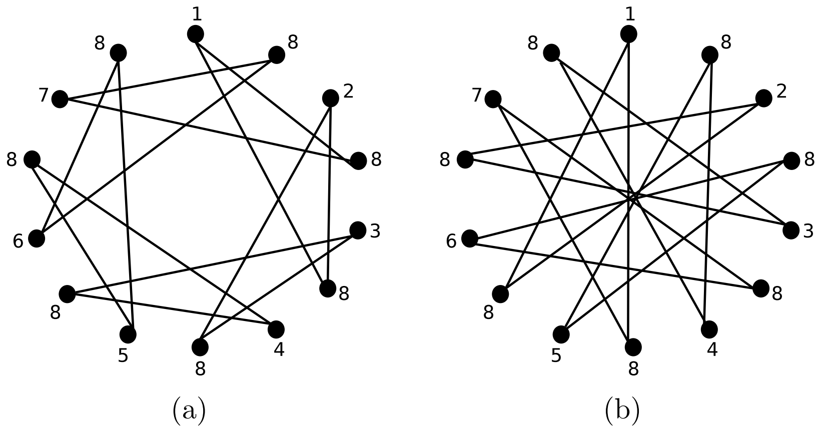


Figure 8. A 8-labeled-packing of three copies of C_{14}: (a) $\sigma^1(C_{14})$, (b) $\sigma^2(C_{14})$.

Subcase 2.2: $(k, m) = (2, 2)$. From Lemma 7, we have $\lambda^2(C_{10}) \le 7$. Figure 9 gives a 7-labeled-embedding of C_{10}.

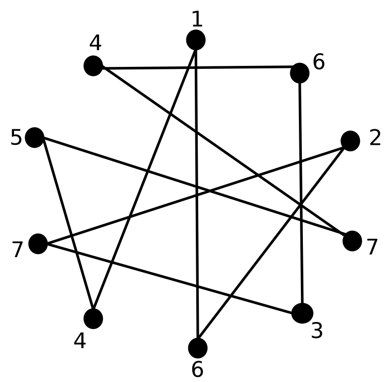


Figure 9. A 7-labeled-embedding of C_{10}

Subcase 2.3: $m \geq 3$

Let $B = \{b_0, b_1, \ldots, b_{p-1}\}$ be a partition of $V(C_n) \setminus \{v_{n-1}, v_n\}$ into p sets, where all sets b_i of B have the same cardinalty $2m$. Then, we have $p = k + \frac{x}{2m}$ such that for $0 \leq i \leq p-1$, $b_i = \{v_1^i, v_2^i, \ldots, v_{2m}^i\}$, where $v_j^i = v_{2im+j}$.

A mapping f from $V(C_n)$ to $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + m\}$ is defined as follows: for every set b_i in B, let

$$
\begin{aligned}
& f(v_{2j+1}^i) = im + j + 1 && \text{for } 0 \leq j \leq m-1. \\
& f(v_{2j}^i) = \tfrac{n}{2} + j && \text{for } 1 \leq j \leq m. \\
& f(v_{n-1}) = \tfrac{n}{2}. \\
& f(v_n) = \tfrac{n}{2} + m - 1.
\end{aligned}
$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \leq t \leq p-1$:

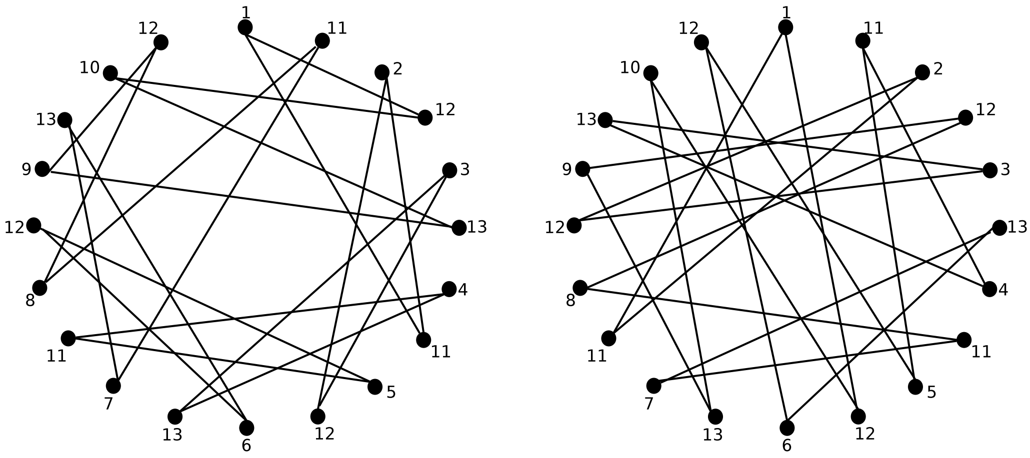
$$
\begin{aligned}
& \sigma^t(v_{2j+1}^i) = v_{2j+1}^i && \text{for all } (i,j) \text{ satisfying } 0 \leq i \leq p-1, 0 \leq j \leq m-1. \\
& \sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod p} && \text{for all } (i,j) \text{ satisfying } 0 \leq i \leq p-1, 1 \leq j \leq m \\
& && \text{and } j \neq m-1. \\
& \sigma^t(v_{2m-2}^i) = v_{2m-2}^{(i+t) \bmod (p+1)} && \text{for } 0 \leq i \leq p-1 \text{ and } i \neq p-t. \\
& \sigma^t(v_{2m-2}^{p-t}) = v_n. \\
& \sigma^t(v_{n-1}) = v_{n-1}. \\
& \sigma^t(v_n) = v_{2m-2}^{t-1}.
\end{aligned}
$$

Using the same proof as in Subcase 1.1, we can show that the mapping f is a $(\lfloor \frac{n}{2} \rfloor + m)$-labeled-packing of k copies of C_n.

To illustrate this case, Figure 10 presents a 13-labeled-packing of three copies of C_{20}. $\square$

3 Labeled packing of cycles of order at most $4k-1$

In the case where n is relatively small compared to k, some additional difficulties arise naturally, and $\lambda^k(C_n)$ has to be estimated differently. We know that any complete graph K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles if



(a) (b)

Figure 10. A 13-labeled-packing of three copies of C_{20}: (a) $\sigma^1(C_{20})$, (b) $\sigma^2(C_{20})$.

n is odd and $\frac{n-2}{2}$ Hamilton cycles plus a perfect matching if n is even. Thus, C_n cannot admit a packing of k copies if $n \leq 2k$. After much work, we raise the following conjecture.

Conjecture 9 *For every cycle C_n of order $n = 2k + x$, where $k \geq 2$ and $1 \leq x \leq 2k - 1$,*

$$\lambda^k(C_n) = \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even.} \\ x + 2 & \text{otherwise .} \end{cases}$$

This conjecture asserts that the upper bound of Lemma 7 is reached for all $n = 4k-1, 4k-2$ and $4k-3$, where $(k, n) \neq (2, 5)$. We report in the following the results of our attempt to give some support to Conjecture 9 for some particular cases.

Theorem 10 *For every cycle C_n of order $n = 2k + x$, where $k \geq 2$, $2k - 3 \leq x \leq 2k - 1$ and $(k, n) \neq (2, 5)$, we have*

$$\lambda^k(C_n) = x + 2$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n)$ be a cycle of order $n = 2k+x$, where $2k-3 \leq x \leq 2k-1$ and $(k, n) \neq (2, 5)$. We know that a maximum independent set of C_n has size $\lfloor \frac{n}{2} \rfloor$. Then, from Lemma 7, we have $\lambda^k(C_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor = x + 2$, it then suffices to prove that C_n admits a $(x + 2)$-labeled packing of k copies. In what follows, we give the proof only for the case $x = 2k-1$, since the proofs of the other cases are similar.

Assume that $x = 2k - 1$ and let $B = \{b_0, b_1, \ldots, b_{k-1}\}$ be a partition of $V(C_n)$ into k sets, such that for $0 \leq i \leq k - 2$, $b_i = \{v_1^i, v_2^i, v_3^i, v_4^i\}$, and

$b_{k-1} = \{v_1^{k-1}, v_2^{k-1}, v_3^{k-1}\}$, where $v_j^i = v_{4i+j}$.

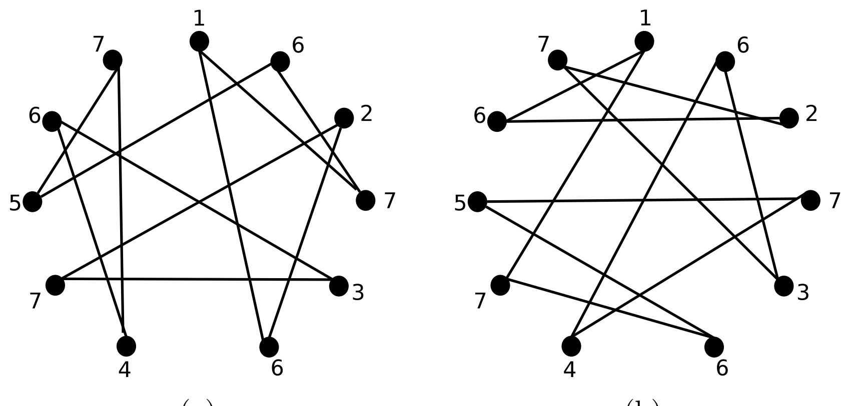
A mapping f from $V(C_n)$ to $\{1, 2, \ldots, 2k+1\}$ is defined as follows:

$$\begin{aligned}
&f(v_{2j+1}^i) = 2i + j + 1 && \text{for all } (i,j) \text{ where } 0 \le j \le 1,\ 0 \le i \le k-2.\\
&f(v_2^i) = 2k && \text{for } 0 \le i \le k-2.\\
&f(v_4^i) = 2k+1 && \text{for } 0 \le i \le k-2.\\
&f(v_1^{k-1}) = 2k-1. &&\\
&f(v_2^{k-1}) = 2k. &&\\
&f(v_3^{k-1}) = 2k+1. &&
\end{aligned}$$

Let $\sigma^0(C_n)$, $\sigma^1(C_n), \ldots, \sigma^{k-1}(C_n)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively, where $\sigma^0(C_n) = C_n$ and for $1 \le t \le k-1$, let

$$\begin{aligned}
&\sigma^t(v_{2j+1}^i) = v_{2j+1}^i && \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-2, 0 \le j \le 1.\\
&\sigma^t(v_{2j}^i) = v_{2j}^{(i+t) \bmod k} && \text{for all } (i,j) \text{ satisfying } 0 \le i \le k-2, 1 \le j \le 2.\\
& && \text{and } (i,j) \ne (k-t-1, 2).\\
&\sigma^t(v_4^{k-t-1}) = v_3^{(k-1)}. &&\\
&\sigma^t(v_1^{k-1}) = v_1^{(k-1)}. &&\\
&\sigma^t(v_2^{k-1}) = v_2^{(t-1)}. &&\\
&\sigma^t(v_3^{k-1}) = v_4^{(t-1)}. &&
\end{aligned}$$

Figure 11 presents a 7-labeled packing of three copies of C_{11}. □



(a) (b)

Figure 11. A 7-labeled packing of three copies of C_{11}: (a) $\sigma^1(C_{11})$, (b) $\sigma^2(C_{11})$.

We now present our results concerning the cycles of order $2k+1$ and $2k+2$.

Theorem 11 *For every prime number $k > 2$, we have $\lambda^k(C_{2k+1}) < 4$.*

Proof. Let $C_{2k+1} = (v_1, v_2, \ldots, v_{2k+1}, v_{2k+1} = v_1)$. We assume that there exists a mapping $f : V(C_{2k+1}) \longrightarrow L = \{1, 2, 3, 4\}$ which is a 4-labeled packing of C_{2k+1}. Let $\sigma^0(C_{2k+1})$, $\sigma^1(C_{2k+1}), \ldots, \sigma^{k-1}(C_{2k+1})$ be a packing of k copies of C_{2k+1} into K_{2k+1} under the permutations $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$, respectively. We consider two cases according to the cardinality of S_f (which is defined in Section 1).

Case 1. $|S_f| = 0$. In this case, each label occurs at least twice in $f(V(C_n))$. Let us introduce the following notation: for every label i of L, let V_i be the subset of vertices with color i. We know that $\sigma^0(C_{2k+1}) \cup \sigma^1(C_{2k+1}) \cup \cdots \cup \sigma^{k-1}(C_{2k+1}) = K_{2k+1}$. We know that all vertices of the complete graph are neighbors. Then, for every label i of L, the number of edges in the subgraph of K_{2k+1} induced by the set V_i is $\frac{|V_i|(|V_i|-1)}{2}$. Then, for every cycle $\sigma^i(C_{2k+1})$ of the packing, we must have exactly $\frac{|V_i|(|V_i|-1)}{2k} = p_i$ edges between the vertices colored with i in the cycle $\sigma^i(C_{2k+1})$. Hence, the following system of equations must have at least one solution.

$$\begin{cases} |V_i|(|V_i| - 1) = 2kp_i \quad \text{for } 1 \leq i \leq 4 & (1) \\ |V_1| + |V_2| + |V_3| + |V_4| = 2k + 1 & (2) \end{cases}$$

Since k is prime, equation (1) has a solution if and only if $|V_i|$ or $|V_i| - 1$ are multiples of k (where $1 \leq i \leq 4$) since k is prime. Then we have $|V_1| + |V_2| + |V_3| + |V_4| > 2k + 1$, therefore this system of equations has no solution which is a contradiction for $k > 2$.

Case 2. $|S_f| \geq 1$. Let v_i be any vertex of $V_{S(f)}$. Then we consider two cases according to the labels of the neighbors vertices of v_i.

Subcase 1. $f(v_{i-1}) = f(v_{i+1}) = c$. In this case, we must have at least $2k$ vertices colored with c. This requires at least $2k + 3$ vertices in C_{2k+1}, which is a contradiction.

Subcase 2. $f(v_{i-1}) \neq f(v_{i+1})$. From hypothesis, each of the two labels $f(v_{i-1})$ and $f(v_{i+1})$ must occur at least k times. This requires at least $2k + 2$ vertices in C_{2k+1}. Hence, we also reached a contradiction and the theorem is proved. $\square$

In the rest of this section, we show a new upper bound for the labeled packing number of some particular cases of C_{2k+1} and C_{2k+2}. Using the same proof as in the above theorem, we can obtain the following results.

Theorem 12 *For every even number $k \geq 2$, where k is a power of 2, we have $\lambda^k(C_{2k+1}) = 2$.*

Proof. Let $C_{2k+1} = v_1, v_2, \ldots, v_{2k+1}, v_{2k+2} = v_1$. We first prove that $\lambda^k(C_{2k+1}) < 3$. We assume that there exists a mapping $f : V(C_{2k+1}) \longrightarrow L = \{1, 2, 3\}$ which is 3-labeled-packing of C_{2k+1}. Let $\sigma^1(C_n)$, $\sigma^2(C_n), \ldots, \sigma^k(C_k)$ be a packing of k copies of C_n into K_n under the permutations $\sigma^1, \sigma^2, \ldots, \sigma^k$, respectively. We consider two cases according to the cardinality of S_f.

Case 1. $|S_f| = 0$**.** Using exactly the same techniques as in the proof of Theorem 11 (Case 1), we can show that $\lambda^k(C_{2k+1}) < 3$.

Case 2. $|S_f| > 1$**.** Let v_i be a vertex of S_f. Then we consider two subcases according to the labels of the neighboring vertices of v_i.

Subcase 1. $f(v_{i-1}) = f(v_{i+1}) = c$. From hypothesis, we must have at least $2k$ vertices colored with c. This requires at least $2k + 2$ vertices in C_{2k+1}, which is a contradiction.

Subcase 2. $f(v_{i-1}) \neq f(v_{i+1})$. Let $S_f = \{1\}$. In this case, it is clear that we cannot construct a 3-labeled-packing of C_n if we use a labeling function different than the one with the following labeling scheme:

(1) one vertex is colored with 1.
(2) k vertices are colored with 2.
(3) k vertices are colored with 3.

We know that $\sigma^1(C_{2k+1}) \cup \sigma^2(C_{2k+1}) \cup \cdots \cup \sigma^k(C_{2k+1}) = K_{2k+1}$. As on the complete graph all vertices are neighbors, the number of edges in the subgraph of K_k induced by V_2 or V_3 is $\frac{k(k-1)}{2}$. Then, for every cycle $\sigma^i(C_{2k+1})$ of the packing, we must have exactly $\frac{k-1}{2}$ edges in the subgraph of $\sigma^i(C_{2k+1})$ induced by V_2 or V_3, which is a contradiction since k is even.

Since we get a contradiction in both cases, we conclude that $\lambda^k(C_{2k+1}) \leq 2$. It then remains to show that $\lambda^k(C_{2k+1}) = 2$. This can be can be easily proved using the following labeling scheme : $f(v_1) = 1$ and $f(v_i) = 2$ for $2 \leq i \leq 2k+2$.

□

Theorem 13 *For every even number $k \geq 2$, we have $\lambda^k(C_{2k+2}) < 5$.*

Proof. Let $C_{2k+1} = v_1, v_2, \ldots, v_{2k+1}, v_{2k+1} = v_1$. We assume that there exists a mapping $f : V(C_{2k+1}) \longrightarrow L = \{1, 2, 3, 4, 5\}$ which is 5-labeled-packing of C_{2k+1}. Let $\sigma^1(C_{2k+1})$, $\sigma^2(C_{2k+1}), \ldots, \sigma^k(C_{2k+1})$ be a packing of k copies of C_{2k+1} in K_{2k+1} under the permutations $\sigma^1, \sigma^2, \ldots, \sigma^k$, respectively. We consider two cases according to the cardinality of S_f.

Case 1. $|S_f| = 0$. In this case, each label occurs at least two times. It suffices to prove that $\lambda^k(c_{2k+2}) < 3$. We first assume that $\lambda^k(c_{2k+2}) \geq 3$. We know that any complete graph K_{2k+2} can be decomposed into k Hamilton cycles plus a perfect matching of size $k+1$, we know that all vertices of the complete graph are neighbors, then we have

(1) For every label i of L, the number of edges in the subgraph of K_{2k+2} induced by $|V_i|$ is equal to $\frac{|V_i|(|V_i|-1)}{2}$.
(2) For every two labels i and j of L, the number of edges (in K_{2k+2}) between two sets of vertices V_i and V_j where $i \neq j$ is equal to $|V_i| \cdot |V_j|$.

Hence, the following system of equations must have at least one solution.

$$\begin{cases} |V_i|(|V_i|-1) - l_{ii} = 2kp_{ii} \quad \text{for } 1 \leq i \leq 3 & (1) \\ |V_i||V_j| - l_{ij} = kp_{ij} \quad \text{for } 1 \leq i \neq j \leq 3 & (2) \\ |V_1| + |V_2| + |V_3| = 2k+2 & (2) \\ \sum_{i,j=1}^{3} l_{ij} = k+1 & (4) \end{cases}$$

From this system of equations, it is clear from equation (3) that the three values $|V_1|$, $|V_2|$ and $|V_3|$ must be even number. Then, each equation of (1) and (2) has a solution if and only if the values of l_{ii} and l_{ij} are even since k is even. This is a contradiction with equation (4). So this system of equations has no solution, which is a contradiction with hypothesis.

Case 2. $|S_f| > 1$. Let v_i be a vertex of S_f. Then we consider two cases according to the labels of the neighboring vertices of v_i.

Subcase 1. $f(v_{i-1}) = f(v_{i+1}) = c$. From hypothesis, we must have at least $2k$ vertices colored with c. This requires at least $2k+4$ vertices in C_{2k+2}, which is a contradiction.

Subcase 2. $f(v_{i-1}) \neq f(v_{i+1})$. From hypothesis, each of the two vertices $f(v_{i-1})$ and $f(v_{i+1})$ must occur at least k times. This requires at least $2k+3$ vertices in C_{2k+2}. Hence, we also reached a contradiction and the theorem is proved. □

Conclusion

In this paper, we have proved the exact value of $\lambda^k(C_n)$ for all $k \geq 2$ and $n \geq 4k-3$. Our results are summarized in the follwoing table.

value of $\boldsymbol{n}$	$\boldsymbol{\lambda^2(C_n)}$
$n \leq 4$	no packing
$n = 5$	2
$n \geq 6$	$\lfloor \frac{3n}{4} \rfloor$
$\boldsymbol{n = 2km + x}$, **where** $\boldsymbol{x < 2k}$ **and** $\boldsymbol{k > 2}$	$\boldsymbol{\lambda^k(C_n)}$
$n \in \{3, 4, \ldots, 2k\}$	no packing
$n \in \{2k+1, 2k+2, \ldots, 4k-4\}$	remains to prove
$n \geq 4k-3$ and ($x \neq 2$ (mod) 4 or $m \geq 3$)	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor$
$x = 2$ (mod) 4 and $m = 2$	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{k} \rfloor - 1$

As a corollary, the exact value of 2-labeled embedding number of cycles is a direct consequence of Theorem 8, Theorem 10 and Theorem 12.

Corollary 14 *Let C_n be a cycle of order at least* 5*. Then*

$$\lambda^2(C_n) = \begin{cases} \lfloor \frac{3n}{4} \rfloor & \textit{if } n > 5. \\ 2 & n = 5. \end{cases}$$

Similarly, from Theorem 8 and Theorem 10, we obtain immediately the exact value of $\lambda^3(C_n)$ for all $n \geq 9$. Then, it follows from Theorem 11 and Theorem 13 that $\lambda^3(C_7) < 4$ and $\lambda^3(C_8) < 5$. The 3-labeled packing and 4-labeled packing of three copies of C_7 and C_8 are shown in Figure 12 and Figure 13, respectively. Hence, we obtain the following corollary.

Corollary 15 *Let C_n be a cycle of order at least* 7*. Then*

$$\lambda^3(C_n) = \begin{cases} 3 & \textit{if } n = 7. \\ 4 & \textit{if } n = 8. \\ 8 & \textit{if } n = 14. \\ \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{3} \rfloor & \textit{otherwise.} \end{cases}$$

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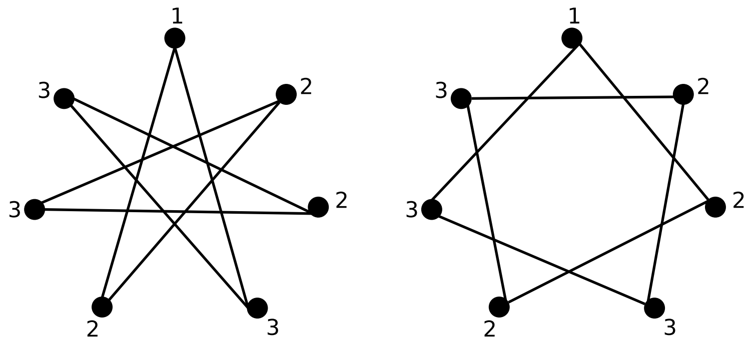
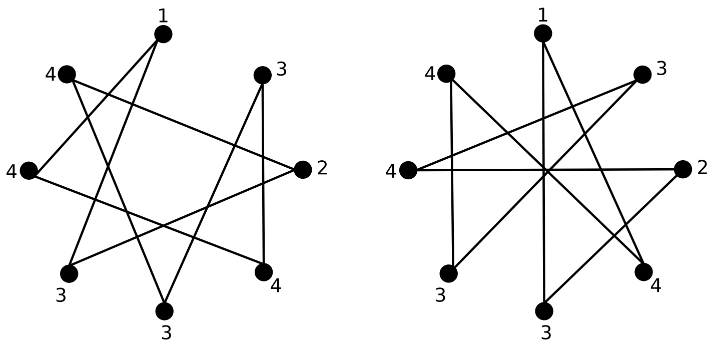


(a) (b)

Figure 12. A 3-labeled packing of three copies of C_7: (a) $\sigma^1(C_7)$, (b) $\sigma^2(C_7)$.



(a) (b)

Figure 13. A 4-labeled packing of three copies of C_8: (a) $\sigma^1(C_8)$, (b) $\sigma^2(C_8)$.

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