Term Algebras, Rewriting

Xavier Urbain

M2IF – Automated deduction

First order terms

Signature: $(\mathcal{S}, \mathcal{F}, \tau)$

- $\mathcal{S}$: set $\neq \emptyset$ of sorts
- $\mathcal{F}$: set of symbols
- τ: function $\mathcal{F} \to \mathcal{S}^{\mathbb{N}_+}$,

$$f \in \mathcal{F} \mapsto s_1 \times \ldots \times s_n \to s$$

n: arity of f

Arity 0: constants

First order terms

$(\mathcal{S}, \mathcal{F}, \tau)$,
$X = \cup_{s \in \mathcal{S}} X_s$: set of variables

$\mathcal{T}(\mathcal{F}, X)$: smallest set such that

- $x \in X_s$ term of sort s
- $f \in \mathcal{F}$, $f : s_1 \times \ldots \times s_n \to s$,
  $t_1 : s_1, \ldots, t_n : s_n$ terms
  then $f(t_1, \ldots, t_n)$ term of sort s

Ground terms: $\mathcal{T}(\mathcal{F}, \emptyset)$

First order subterms

Terms seen as trees $\rightsquigarrow$ positions

Subterm of t at position p, $t|_p$, defined by set of positions:
$\{q \in \mathbb{N}_+^* \mid p \cdot q \in \mathcal{P}os(t)\} \qquad t|_p(q) = t(p \cdot q)$

Subterm relation: $t \rhd s \quad$ if $\exists p \neq \Lambda$ such that $t|_p = s \qquad$ (proper subterm)

For $t|_p$ $(p \in \mathcal{P}os(t))$ and u of same sort, Replacement $t[u]_p$, defined by:
$\{q \in \mathbb{N}_+^* \mid q \in \mathcal{P}os(t) \bigwedge p \not\leq_{\text{pref.}} q\} \cup \{p \cdot q \mid q \in \mathcal{P}os(u)\}$

$$t[u]_p(q) = t(q) \qquad \text{if } q \in \mathcal{P}os(t) \bigwedge p \not\leq_{\text{pref.}} q$$
$$t[u]_p(p \cdot q) = u(q)$$

First order substitutions

Substitution: application $X \to \mathcal{T}(\mathcal{F}, X)$ respecting sorts

Usually: identity except on a finite set

Notation postfix: $\sigma(x) \leadsto x\sigma$

Extended to terms by unique $H_\sigma : \mathcal{T}(\mathcal{F}, X) \to \mathcal{T}(\mathcal{F}, X)$

- $H_\sigma(x) = x$ if x not in σ's domain
- $H_\sigma(x) = x\sigma$ if x in σ's domain
- $H_\sigma(f(s_1, \ldots, s_m)) = f(H_\sigma(s_1), \ldots, H_\sigma(s_m))$ if $f(s_1, \ldots, s_m) \in \mathcal{T}(\mathcal{F}, X)$

Renaming: equivalence relation

akin to α-conversion

$\sigma = \{x_1 \mapsto y_1, \cdots, x_n \mapsto y_n\}$ y_i pairwise distincts

First order substitutions

Abuse: substitution $\leadsto$ extended substitution

Matching: for s t terms, finding σ such that $s\sigma = t$

Unification: for s t terms, finding σ such that $s\sigma = t\sigma$

- Decidable (1st order, e.g. Robinson)
- If unifiable: unique most general unifier (vanilla)

A hint of semantics $\mathcal{F}$-algebra

For $(\mathcal{S}, \mathcal{F}, \tau)$ a signature, $\mathcal{F}$-algebra:

- Support $A_s \neq \emptyset$ for each $s \in \mathcal{S}$
- Application $f_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$
  for all $f \in \mathcal{F}, f : s_1 \times \cdots \times s_n \to s$

$\mathcal{T}(\mathcal{F}, X)$ $\mathcal{F}$-algebra

For A B $\mathcal{F}$-algebras, homomorphism from A to B:
set of applications $h_s : A_s \to B_s$ such that for all $f : s_1 \times \cdots s_n \to s$

$$\forall a_1, \cdots, a_n \in A, \quad h_s(f_A(a_1, \cdots, a_n)) = f_B(h_{s_1}(a_1), \cdots, h_{s_n}(a_n))$$

A-assignment: homomorphism from $\mathcal{T}(\mathcal{F}, X)$ to A

Congruence (equi. relation compatible with A) $\leadsto$ Quotient

A hint of semantics $\mathcal{F}$-algebra

Equation: pair of terms of same sort, $s = t$

set of equations E

Model: A $\mathcal{F}$-algebra, $A \models E$ if $\forall s = t \in E$, $\forall A$-assignment σ, $s\sigma = t\sigma$

$=_E$ smallest congruence on $\mathcal{T}(\mathcal{F}, X)$ such that $\forall \sigma, \forall s = t \in E, s\sigma =_E t\sigma$

$s = t$ an equation, word problem related to $s = t$: $E \models^? s = t$

To solve it: equational reasoning

Starting from E, uses of: *Reflexivity, Symmetry, Transitivity, Replacement*
If derivation: $E \vdash s = t$

[(Birkoff)] if at least a ground term per sort,
$E \vdash s = t \quad \Leftrightarrow \quad E \models s = t \quad \Leftrightarrow \quad s =_E t$

Term Rewriting system, relation

Rewriting rule: couple of terms $s \to t$ (oriented equation)

$$s \xrightarrow[l\to r,\sigma]{p} t \quad \text{if} \quad s|_p \equiv l\sigma \quad t \equiv s[r\sigma]_p$$

Rewriting system: set of rules

Relation $\underset{R}{\to}$: $s \underset{R}{\to} t$ iff $\exists l \to r \in R, \exists p \in \mathcal{P}os(s), \exists \sigma, s \xrightarrow[l\to r,\sigma]{p} t$

Actually, system extension by *monotony* and *stability*

Reflexive/transitive closure: $\underset{R}{\to}^{\star}$ Transitive closure: $\underset{R}{\to}^{+}$

Converse: $\underset{R}{\leftarrow}$ Reflexive/symmetric/transitive: $\overset{\star}{\underset{R}{\leftrightarrow}}$

Term Rewriting example

Binary arithmetics Encoding of $6 : \#110$

- Variables: $X = \{x \; ; \; y \cdots\}$
- Signature: $\mathcal{F} = \{\# \; ; \; 0 \; ; \; 1 \; ; \; +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

Relation $\to$ monotone: if $s \to t$ then $C[s] \to C[t]$

stable: if $s \to t$ then $s\sigma \to t\sigma$

Term Rewriting example

Binary arithmetics Encoding of $6 : \#110$

- Variables: $X = \{x \; ; \; y \cdots\}$
- Signature: $\mathcal{F} = \{\# \; ; \; 0 \; ; \; 1 \; ; \; +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

$$\#10 + \#1$$

Term Rewriting example

Binary arithmetics Encoding of $6 : \#110$

- Variables: $X = \{x \; ; \; y \cdots\}$
- Signature: $\mathcal{F} = \{\# \; ; \; 0 \; ; \; 1 \; ; \; +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

$$\#10 + \#1 \to (\#1 + \#)1$$

Term Rewriting — example

Binary arithmetics — Encoding of 6 : $\#110$

- Variables: $X = \{x \,;\, y \cdots\}$
- Signature: $\mathcal{F} = \{\# \,;\, 0 \,;\, 1 \,;\, +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\left\{\begin{array}{lcllcl} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

$$\#10 + \#1 \to (\#1 + \#)1 \to (\#1)1$$

Term Rewriting — example

Binary arithmetics — Encoding of 6 : $\#110$

- Variables: $X = \{x \,;\, y \cdots\}$
- Signature: $\mathcal{F} = \{\# \,;\, 0 \,;\, 1 \,;\, +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\left\{\begin{array}{lcllcl} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

$$\#10 + \#1 \to (\#1 + \#)1 \to \#11$$ Stop

Term Rewriting — properties

Computing power: Turing complete

Questions:

- Existence of result $\rightsquigarrow$ termination
- Unicity of result $\rightsquigarrow$ confluence, convergence

Term Rewriting — termination

Questions:

- Existence of result $\rightsquigarrow$ termination
- Unicity of result $\rightsquigarrow$ confluence, convergence

t normal form for R: no u such that $t \underset{R}{\to} u$

t normal form of s for R: no u such that $t \underset{R}{\to} u$ and $s \underset{R}{\to}^{\star} t$

$\rightsquigarrow$ s normalisable

System normalising: every term normalisable

System strongly normalising: every derivation $\rightsquigarrow$ normal form
(every term accessible for $\underset{R}{\to}$)

Term Rewriting confluence

Questions:

- Existence of result ⇝ terminaison
- Unicity of result ⇝ confluence, convergence

R Church-Rosser: $u \overset{\star}{\longleftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

R confluent: $u \leftarrow^\star s \rightarrow^\star v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

R locally confluent: $u \leftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

Term Rewriting confluence

Example.

$$\begin{array}{lcl} x + 0 & \rightarrow & x \\ x + x^{-1} & \rightarrow & 0 \\ (x+y)+z & \rightarrow & x+(y+z) \end{array}$$

$$x + (x^{-1} + z) \ \leftarrow\ (x + x^{-1}) + z \ \rightarrow\ 0 + z$$

Not confluent

Term Rewriting confluence

Questions:

- Existence of result ⇝ terminaison
- Unicity of result ⇝ confluence, convergence

R Church-Rosser: $u \overset{\star}{\longleftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

$\Updownarrow ?$

R confluent: $u \leftarrow^\star s \rightarrow^\star v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

$\Updownarrow ?$

R locally confluent: $u \leftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

Term Rewriting confluence

R Church-Rosser: $u \overset{\star}{\longleftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

$\Updownarrow$

R confluent: $u \leftarrow^\star s \rightarrow^\star v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

$\cancel{\Updownarrow}$

R locally confluent: $u \leftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

Theorem. (Lemme de Newman)
Local confluence ⇔ confluence for strongly normalising relations

Term Rewriting confluence

Decidability

Given: finite system R
Question: R confluent?

Undecidable (red. word pb.)

Term Rewriting confluence

Critical pair : $r\rho\sigma = (l\rho[d]_p)\sigma$ where

- $l \to r \in R, \quad g \to d \in R$
- p position non variable of l
- ρ renaming of l
- σ most general unifier of $l|_p$ and g (rule superposition)

Set of critical pairs of R: $\mathrm{CP}(R)$

Theorem.
Local confluence of pairs of $\mathrm{CP}(R)$ ⇔ locale confluence of R

⇝ Decidable for strongly normalising finite systems

Termination overview

Fundamental property:

- Inductions,
- Totality of functions,
- Preliminary,
- Selfstabilisation, liveness, etc.

In this lecture: first order term rewriting
Enough? Turing complete (cfr poly)

Termination overview

Automation? ⇝ correct, incomplete. . .

Always difficult

- $f(f(x)) \to f(x)$
- $f(a, b, x) \to f(x, x, x)$

$$\begin{array}{rcl}
\mathsf{ef}(x)[y]_{\mathtt{t}} & \rightarrow & \mathsf{ef}(x[y]_{\mathtt{t}}) \\
\mathsf{Pe}(x)[y]_{\mathtt{f}} & \rightarrow & \mathsf{Pe}(x[y]_{\mathtt{t}}) \\
(f \vee g)[s]_{\mathtt{f}} & \rightarrow & (f[s]_{\mathtt{f}}) \vee (g[s]_{\mathtt{f}}) \\
\neg(f)[s]_{\mathtt{f}} & \rightarrow & \neg(f[s]_{\mathtt{f}}) \\
(f \wedge g)[s]_{\mathtt{f}} & \rightarrow & (f[s]_{\mathtt{f}}) \wedge (g[s]_{\mathtt{f}}) \\
x[i]_{\mathtt{t}}\, d & \rightarrow & x \\
(f \Rightarrow g)[s]_{\mathtt{f}} & \rightarrow & (f[s]_{\mathtt{f}}) \Rightarrow (g[s]_{\mathtt{f}}) \\
1[x \cdot s]_{\mathtt{t}} & \rightarrow & x \\
\exists(f)[s]_{\mathtt{f}} & \rightarrow & \exists(f[1 \cdot (s \circ \uparrow)]_{\mathtt{f}}) \\
f[i]_{\mathtt{f}}\, d & \rightarrow & f \\
\forall(f)[s]_{\mathtt{f}} & \rightarrow & \forall(f[1 \cdot (s \circ \uparrow)]_{\mathtt{f}}) \\
(f[s]_{\mathtt{f}})[t]_{\mathtt{f}} & \rightarrow & f[s \circ t]_{\mathtt{f}} \\
(x[s]_{\mathtt{t}})[t]_{\mathtt{t}} & \rightarrow & x[s \circ t]_{\mathtt{t}} \\
id \circ s & \rightarrow & s \\
\uparrow \circ (x \cdot s) & \rightarrow & s \\
(s \circ t) \circ u & \rightarrow & s \circ (t \circ u) \\
(x \cdot s) \circ t & \rightarrow & (x[t]_{\mathtt{t}}) \cdot (s \circ t) \\
s \circ id & \rightarrow & s \\
1 \cdot \uparrow & \rightarrow & id \\
(1[s]_{\mathtt{t}}) \cdot (\uparrow \circ s) & \rightarrow & s \\
a, \triangledown & \rightarrow & a \\
a, a & \rightarrow & a \\
a \bullet \lozenge & \rightarrow & a \\
a \bullet a & \rightarrow & a \\
\neg(\neg(f)) & \rightarrow & f \\
f \wedge f & \rightarrow & f
\end{array}$$

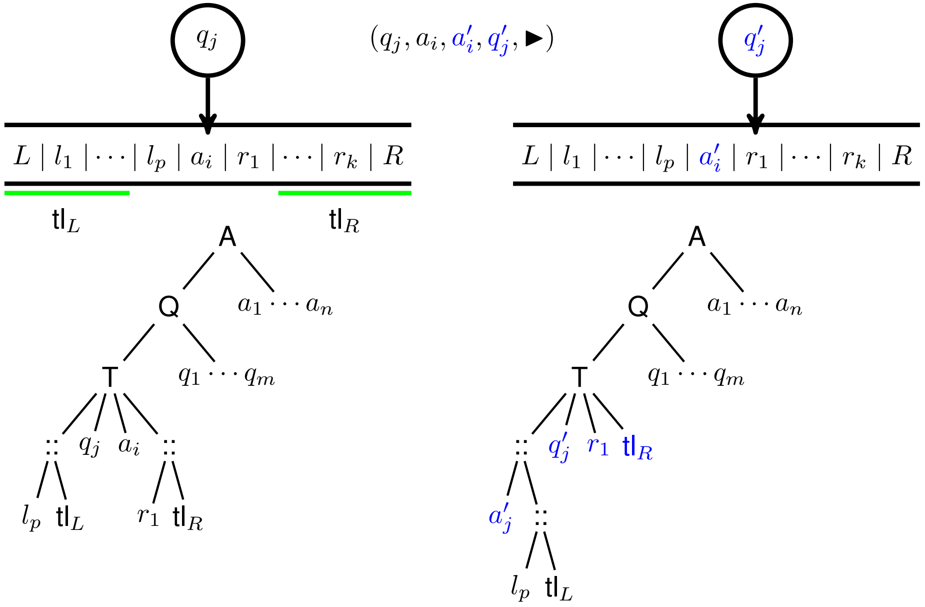
$$\begin{array}{rcl}
f \vee f & \rightarrow & f \\
f \Rightarrow g & \rightarrow & \neg(f) \vee g \\
\exists(f) & \rightarrow & \neg(\forall(\neg(f))) \\
(a, \{_{\mathtt{f}} \neg(f)\}) \vdash b & \rightarrow & a \vdash (\{_{\mathtt{f}} f\}, b) \\
\{_{\mathtt{f}} \neg(f)\} \vdash b & \rightarrow & \triangledown \vdash (b, \{_{\mathtt{f}} f\}) \\
a \vdash (\{_{\mathtt{f}} \neg(f)\}, b) & \rightarrow & (a, \{_{\mathtt{f}} f\}) \vdash b \\
a \vdash \{_{\mathtt{f}} \neg(f)\} & \rightarrow & (a, \{_{\mathtt{f}} f\}) \vdash \triangledown \\
(a, \{_{\mathtt{f}} f \wedge g\}) \vdash b & \rightarrow & (a, \{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}) \vdash b \\
\{_{\mathtt{f}} f \wedge g\} \vdash b & \rightarrow & (\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}) \vdash b \\
a \vdash (\{_{\mathtt{f}} f \vee g\}, b) & \rightarrow & a \vdash (\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}, b) \\
a \vdash \{_{\mathtt{f}} f \vee g\} & \rightarrow & a \vdash (\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}) \\
\{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} f \wedge g\}, b)\} & \rightarrow & \{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} f\}, b)\} \bullet \{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} g\}, b)\} \\
\{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} f \wedge g\}\} & \rightarrow & \{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} f\}\} \bullet \{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} g\}\} \\
\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f \vee g\}) \vdash b\} & \rightarrow & \{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash b\} \bullet \{_{\mathtt{s}} (a, \{_{\mathtt{f}} g\}) \vdash b\} \\
\{_{\mathtt{s}} \{_{\mathtt{f}} f \vee g\} \vdash b\} & \rightarrow & \{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash b\} \bullet \{_{\mathtt{s}} \{_{\mathtt{f}} g\} \vdash b\} \\
\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash (\{_{\mathtt{f}} f\}, b)\} & \rightarrow & \lozenge \\
\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash \{_{\mathtt{f}} f\}\} & \rightarrow & \lozenge \\
\{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash (b, \{_{\mathtt{f}} f\})\} & \rightarrow & \lozenge \\
\{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash \{_{\mathtt{f}} f\}\} & \rightarrow & \lozenge \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} (a, f) \vdash (g, b)\} & \rightarrow & \{_{\mathtt{s}} a \vdash b\} \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} (a, f) \vdash b\} & \rightarrow & \{_{\mathtt{s}} a \vdash b\} \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} a \vdash (b, f)\} & \rightarrow & \{_{\mathtt{s}} a \vdash b\} \\
\{_{\mathtt{s}} a \vdash \triangledown\} \bullet \{_{\mathtt{s}} (a, f) \vdash b\} & \rightarrow & \{_{\mathtt{s}} a \vdash \triangledown\} \\
\{_{\mathtt{s}} a \vdash (b, f)\} \bullet \{_{\mathtt{s}} \triangledown \vdash b\} & \rightarrow & \{_{\mathtt{s}} \triangledown \vdash b\} \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} \triangledown \vdash b\} & \rightarrow & \{_{\mathtt{s}} \triangledown \vdash b\} \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} a \vdash \triangledown\} & \rightarrow & \{_{\mathtt{s}} a \vdash \triangledown\} \\
\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} \triangledown \vdash \triangledown\} & \rightarrow & \{_{\mathtt{s}} \triangledown \vdash \triangledown\}
\end{array}$$

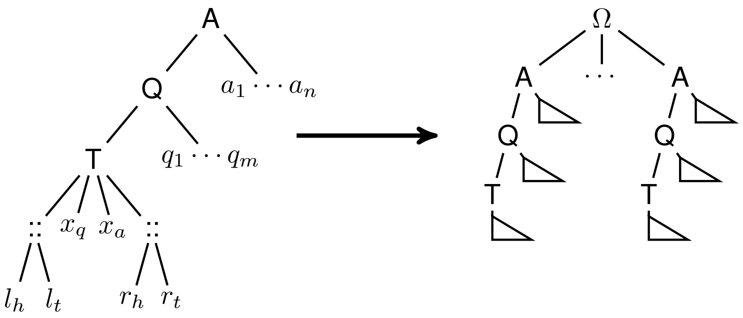
Termination overview

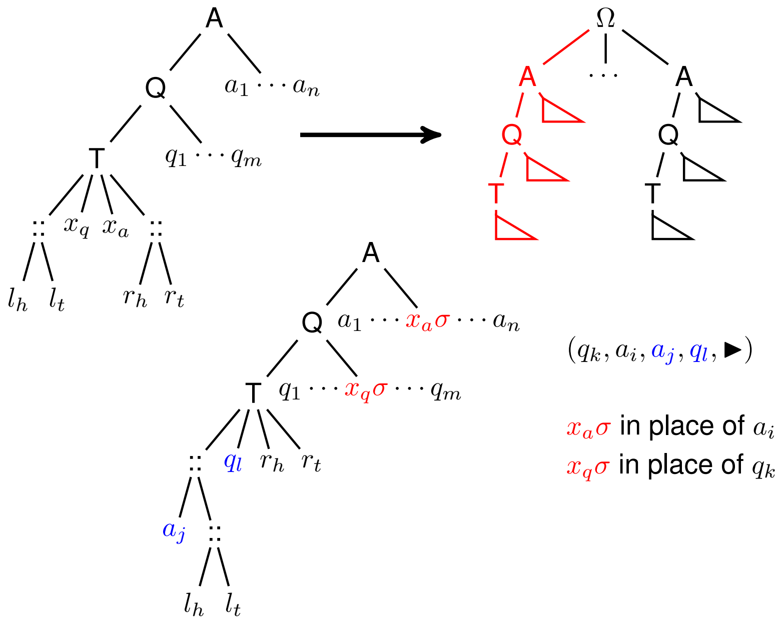
Automation? ⇝ correct, incomplete...

Always difficult

- $f(f(x)) \rightarrow f(x)$
- $f(a, b, x) \rightarrow f(x, x, x)$
- > 50 rules + equational theories
- > 1800 rules (> 1000 symbols)
- $\begin{cases} a(a(x)) & \rightarrow & b(c(x)) \\ b(b(x)) & \rightarrow & a(c(x)) \\ c(c(x)) & \rightarrow & a(b(x)) \end{cases}$
- Syracuse...



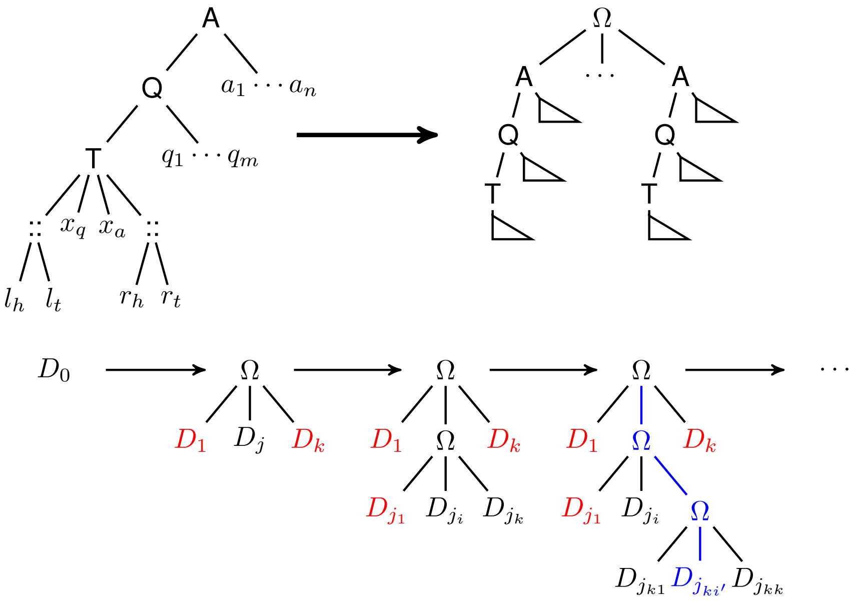




$(q_k, a_i, a_j, q_l, \blacktriangleright)$

$x_a\sigma$ in place of a_i

$x_q\sigma$ in place of q_k



Termination proof

Termination of TRS = well foundness of relation (WF) ⇝ « measure »

How? $\mathcal{R} \subseteq O \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$

$\mathcal{R}, O, f$ such that $s\ \mathcal{R}^+\ t \Rightarrow f(s)\ O^+\ f(t) \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$.

$(\text{WF}(O) \Rightarrow \text{WF}(O^+) \Rightarrow \text{WF}(O^+ \circ f) \Rightarrow \text{WF}(\mathcal{R}^+))$

Problem transformation until: pb. trivial, pb. inclusion

$$\textsc{Rule name}(\textsc{param})\ \frac{p_1 \dots p_n}{p}\ \textsc{conditions}$$

Transf. correct and complete: criterion ⇝ termination constraints

Inclusion: ordering constraints

Termination proof

Termination of TRS = well foundness of relation (WF) ⇝ « measure »

How? $\mathcal{R} \subseteq O \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$

$\mathcal{R}, O, f$ such that $s\ \mathcal{R}^+\ t \Rightarrow f(s)\ O^+\ f(t) \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$.

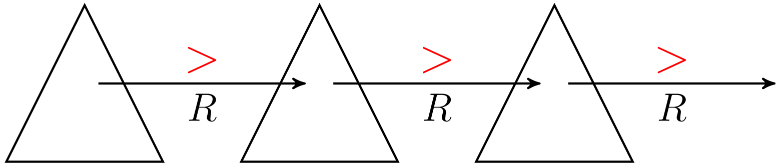
$(\text{WF}(O) \Rightarrow \text{WF}(O^+) \Rightarrow \text{WF}(O^+ \circ f) \Rightarrow \text{WF}(\mathcal{R}^+))$

Problem transformation until: pb. trivial, pb. inclusion

$$\dfrac{\dfrac{\dfrac{<_1 \text{ (poly. interp.)}}{\{\dots\langle u'_n, v'_n\rangle\dots\}\text{ SN?}}\ < \quad \dfrac{\dfrac{<_2 \text{ (RPO)}}{\{\dots\langle u''_k, v''_k\rangle\dots\}\text{ SN?}}\ < \quad \dfrac{}{\{\dots\langle u''_j, v''_j\rangle\dots\}\text{ SN?}}}{\{\dots\langle u'_i, v'_i\rangle\dots\}\text{ SN?}}\ \textsc{Graph}}{\mathcal{R}_{\text{DP}} = \{\dots\langle u_i, v_i\rangle\dots\}\text{ SN?}}\ \textsc{Graph}}{\mathcal{R}_{\text{init}} = \{\dots l_i \to r_i \dots\}\text{ SN?}}\ \text{DP}$$

Termination proof

Inclusion $\rightarrow_R \subseteq >$: $s > t$ for all $s \rightarrow_R t$



Infinitely many $s \rightarrow t \rightsquigarrow$ automation?

Test finite?

Termination Manna-Ness

Idea: ordering on rules stable through closures giving relation

Theorem. (Lankford)

- R a TRS $\{l_1 \rightarrow r_1, \ldots, l_n \rightarrow r_n\}$,
- $<$ such that WF($<$), $<$ stable and monotone,

then $(\forall i, l_i > r_i) \Rightarrow$ SN($\rightarrow_R$).

Ex.

$$\left\{ \begin{array}{lcl} x+0 & \rightarrow & x \\ x+s(y) & \rightarrow & s(x+y) \end{array} \right\} \quad [\![s]\!] >_e [\![t]\!] \text{ with } \begin{array}{lcl} [\![0]\!] & = & 1 \\ [\![s]\!](x) & = & x+1 \\ [\![+]\!](x,y) & = & x+2y \end{array}$$

$$[\![x+0]\!] = x+2 >_e [\![x]\!] = x$$
$$[\![x+s(y)]\!] = x+2y+2 >_e [\![s(x+y)]\!] = x+2y+1$$

Termination Manna-Ness

Idea: ordering on rules stable through closures giving relation

Theorem. (Lankford)

- R a TRS $\{l_1 \rightarrow r_1, \ldots, l_n \rightarrow r_n\}$,
- $<$ such that WF($<$), $<$ stable and monotone,

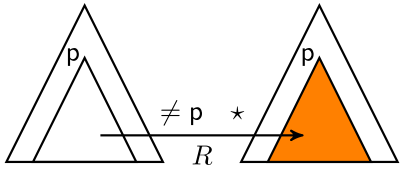
then $(\forall i, l_i > r_i) \Rightarrow$ SN($\rightarrow_R$).

Heavy constraints:

- Complexity ($\rightsquigarrow$ relation),
- Orderings

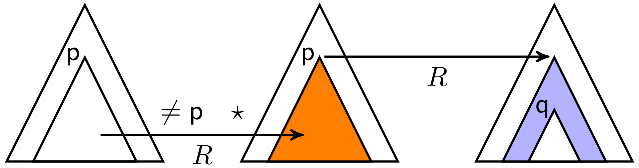
In practice: simplification orderings (easier for WF)
$f(f(x)) \rightarrow f(g(f(x)))$? stuck!

Termination DP



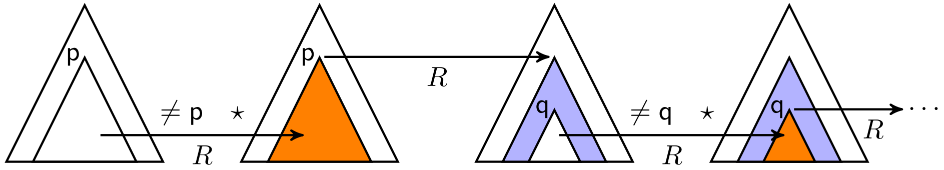
Termination

DP



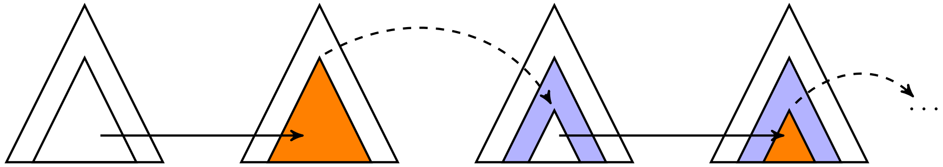
Termination

DP



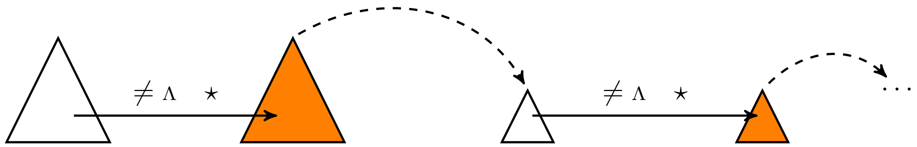
Termination

DP



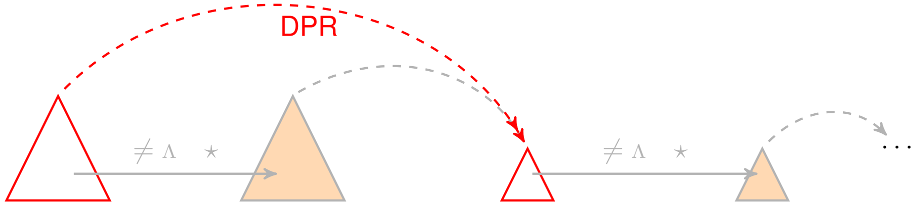
Termination

DP



Termination

DP


DPR
$\neq \Lambda$ $\star$
$\neq \Lambda$ $\star$
...