A New Lattice-based Plane-probing Algorithm

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Abstract

Plane-probing algorithms are fundamental tools to locally capture arithmetical and geometrical properties of digital surfaces (boundaries of a connected set of voxels), and especially normal vector information. On a digital plane, we consider a local pattern, a triangle, that is expanded starting from a point of interest using simple probes of the digital plane with a predicate "Is a point x in the digital plane?". Here, we present a new plane-probing algorithm that is theoretically correct on digital planes, and with better experimental compactness and locality than existing solutions.

Objectives

To design an algorithm that terminates on a triangle which:

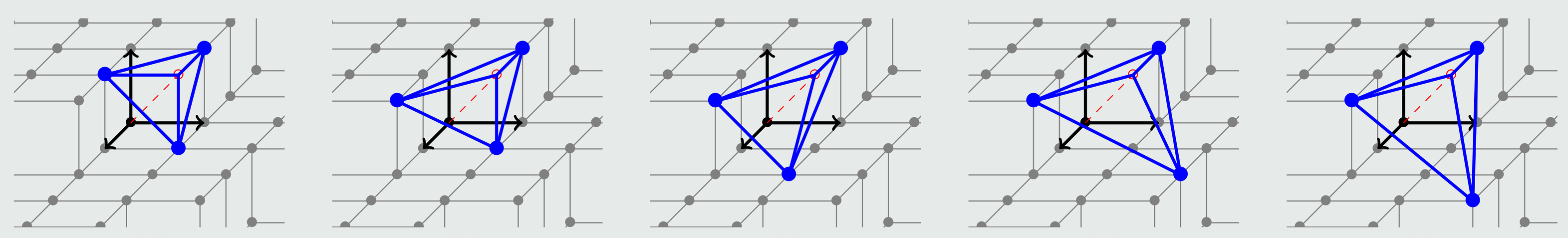
- has normal vector that matches with the expected one for digital plane (correctness).
- be as compact as possible (acute or right angles).
- probes as close as possible to the source point (proximity property).
- minimizes the number of iterations or probes during the computations.

Digital plane

A *digital plane* is an infinite digital set defined by a normal $\mathbf{N} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ as follows[4]:

$$\mathbf{P_N} := \{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \mathbf{x} \cdot \mathbf{N} < \|\mathbf{N}\|_1\}. \quad (1)$$

The evolution of a plane-probing algorithm on a digital plane of normal $(1, 2, 5)$.



(a) (b) (c) (d) (e)

Plane-probing Algorithms

Given a digital plane $\mathbf{P}$, of unknown normal vector $\mathbf{N}$. At every step i, the algorithms update one vertex of $\mathbf{T}^{(i)} = (\mathbf{v}_k^{(i)})_{k \in \mathbb{Z}/3\mathbb{Z}}$. That vertex is replaced by a point of $\mathbf{P}$ from a *neighborhood* in [3]. The three variants H, R and L consider different neighborhoods.

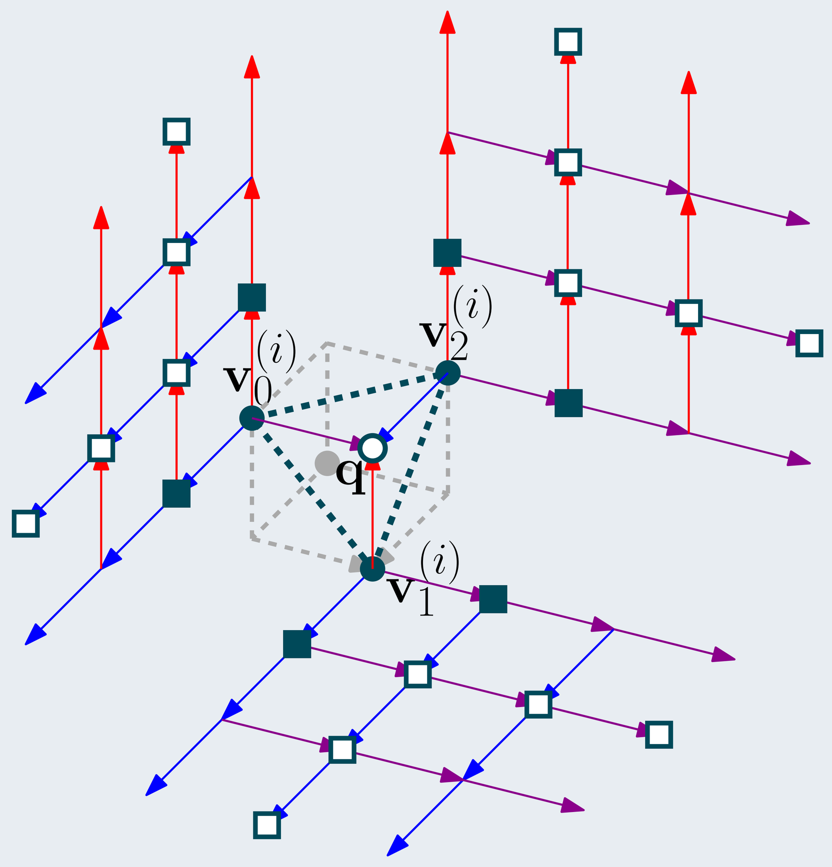


Figure 1. Illustrations of the neighborhoods: $\mathcal{N}_{S_H}^{(i)}$ (■), $\mathcal{N}_{S_R}^{(i)}$ (□) and $\mathcal{N}_{S_L}^{(i)}$ includes every point on the lattices, excepted the triangle vertices.

The algorithms terminate at a step n, when $\mathcal{N}_S^{(n)} \bigcap \mathbf{P} = \emptyset$.

Comparaison between H, R[1] and L

	H	R[1]	L
Cardinal of neighborhood	$\lvert\mathcal{N}_{S_H}\rvert = 6$	$\lvert\mathcal{N}_{S_R}\rvert > 6$	$\lvert\mathcal{N}_{S_L}\rvert > \lvert\mathcal{N}_{S_R}\rvert$
Last triangle is acute	✗	✓(expt.)	✓[1]
The open circumscribing ball that passes the vertices of two consecutive triangle does not include any in-plane point	✗	✗	✓[1]

A total preorder

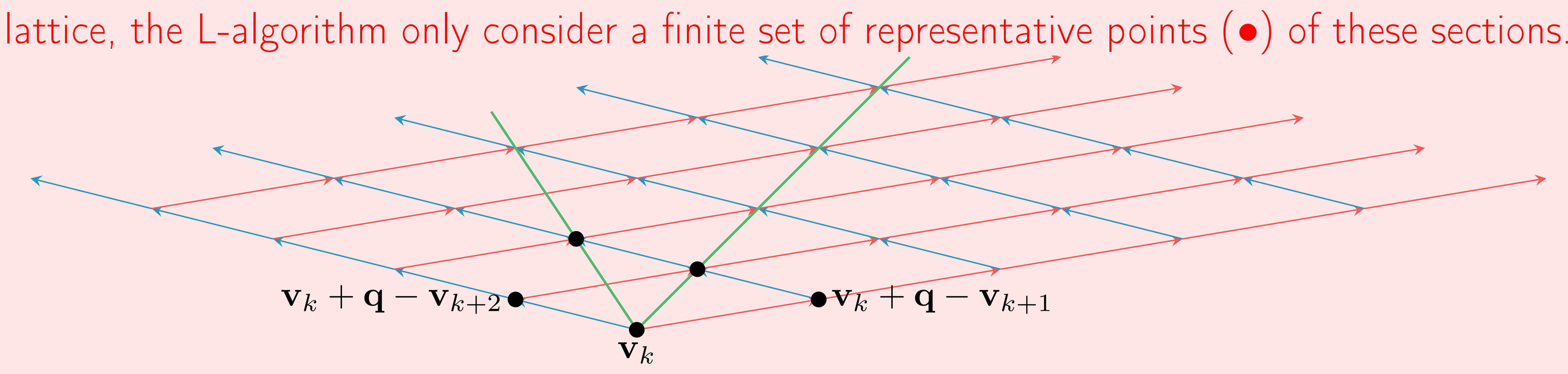
Let $\mathcal{H}_+^{(i)}$ be the half-space delimited by $\mathbf{T}^{(i)}$ and containing $\mathcal{N}_S^{(i)}$. In addition, let $\mathcal{B}(\mathbf{T}, \mathbf{x})$ be the closed ball defined by $\mathbf{T}^{(i)}$ and a fourth point $\mathbf{x}$ not in the plane passing by $\mathbf{T}^{(i)}$. For any pair of points $\mathbf{x}, \mathbf{x}'$, not in the plane passing by $\mathbf{T}^{(i)}$, we say that $\mathbf{x}'$ is *closer* to $\mathbf{T}^{(i)}$ than $\mathbf{x}$ [2], denoted $\mathbf{x}' \leq_{\mathbf{T}^{(i)}} \mathbf{x}$, if and only if $(\mathcal{B}(\mathbf{T}^{(i)}, \mathbf{x}') \cap \mathcal{H}_+^{(i)}) \subseteq (\mathcal{B}(\mathbf{T}^{(i)}, \mathbf{x}) \cap \mathcal{H}_+^{(i)})$.

Consequence of an acute angle

For all $k \in \mathbb{Z}/3\mathbb{Z}$, let Λ_k be the set $\{\mathbf{v}_k + \alpha\mathbf{u} + \beta\mathbf{w} \mid (\alpha, \beta) \in S_L\}$, where $\mathbf{u}$, $\mathbf{w}$ are any two non-zero vectors of $\mathbb{Z}^3$ such that $\mathbf{v}_k + \mathbf{u}, \mathbf{v}_k + \mathbf{w} \in \mathcal{H}_+$. If $\mathbf{u} \cdot \mathbf{w} \geq 0$, we have either $\mathbf{v}_k + \mathbf{u} \leq_{\mathbf{T}} \mathbf{x}$ for all $\mathbf{x} \in \Lambda_k$ or $\mathbf{v}_k + \mathbf{w} \leq_{\mathbf{T}} \mathbf{x}$ for all $\mathbf{x} \in \Lambda_k$.

Smaller candidate set

We partition the neighborhood into sectors with acute angle. Instead of exploring an infinite lattice, the L-algorithm only consider a finite set of representative points (●) of these sections.



Sequence of vectors that defines a candidate set

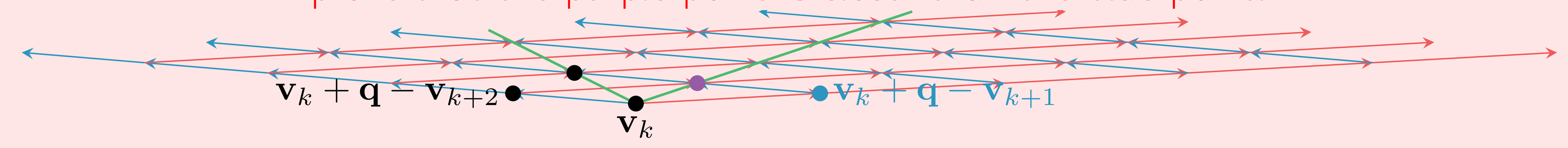
For any $(\mathbf{u}, \mathbf{w}) \in \mathbb{Z}_*^{3\times 2}$, we define a sequence of vector pairs $\Omega_{\mathbf{u},\mathbf{w}} = \{(\mathbf{u}_j, \mathbf{w}_j)\}_{j \geq 0}$:

1. $\mathbf{u}_0 = \mathbf{u}$ and $\mathbf{w}_0 = \mathbf{w}$.
2. For any $j \geq 0$, the pair $(\mathbf{u}_{j+1}, \mathbf{w}_{j+1})$ exists if and only if there exists $\gamma_j \geq 1$ such that $(\mathbf{u}_j + \gamma_j\mathbf{w}_j) \cdot (\mathbf{u}_j + (\gamma_j + 1)\mathbf{w}_j) < 0$, then $\mathbf{u}_{j+1} = \mathbf{w}_j$, $\mathbf{w}_{j+1} = \mathbf{u}_j + \gamma_j\mathbf{w}_j$.

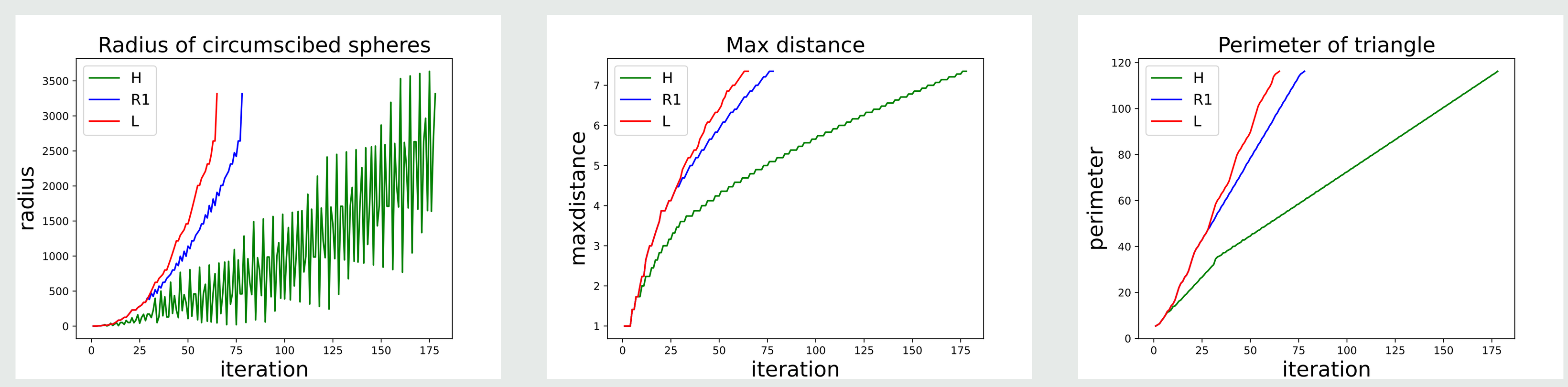
From one vertex $\mathbf{v}_k$, we can define by recurrence the set of candidate points from $\{\mathbf{v}_k + \mathbf{x} | \mathbf{x} \in \Omega_{\mathbf{q}-\mathbf{v}_{k+1},\mathbf{q}-\mathbf{v}_{k+2}}\}$. or $\{\mathbf{v}_k + \mathbf{x} | \mathbf{x} \in \Omega_{\mathbf{q}-\mathbf{v}_{k+2},\mathbf{q}-\mathbf{v}_{k+1}}\}$.

Even smaller candidate set

If there exists an obtuse angle among the angles defined from two consecutive vectors of $\{\mathbf{u} + \gamma\mathbf{w}\}_\gamma$, we don't need to examine every point in $\{\mathbf{v}_k + \mathbf{u} + \gamma\mathbf{w}\}_\gamma$. For example, we can prove that the purple point is closer than the blue point.

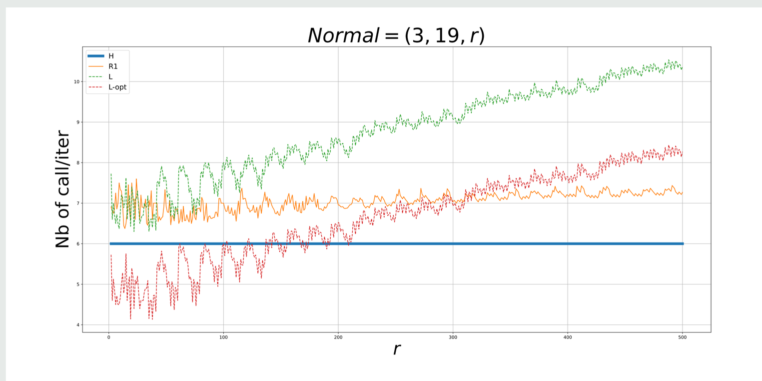


Geometrical property of the L-algorithm



For $\mathbf{N} = (198, 195, 193)$, (from left to right) we measure the radius of circumspheres of two consecutive triangles, maximal distance to $\mathbf{q}$, and perimeters of the triangles.

Complexity: Number of calls to predicate per iteration



Conclusion

We present a new plane-probing algorithm, the L-algorithm, that considers more candidate points per step and requires fewer steps than its predecessors. Even though it examines more points to find the closest one, the selected point provides more interesting compactness features at every step. The circumspheres of consecutive triangles has non-decreasing radii and do not include any point in the plane. In the future, we wish that this property provides better proximity results for plane-probing algorithms.

References

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