Pre-Grundy Games

Games And Graphs Workshop 2017

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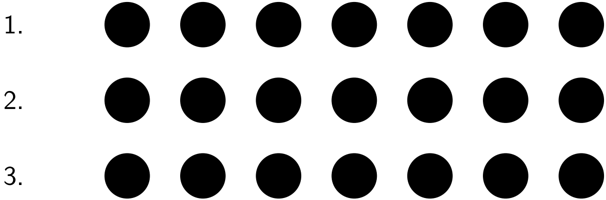
Gabrielle Paris

Are Pre-Grundy games boring ?

Sommaire

Definition

Octal games: [Winning Ways]
A game played on heaps where each player:



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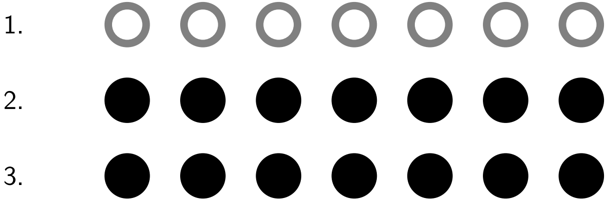

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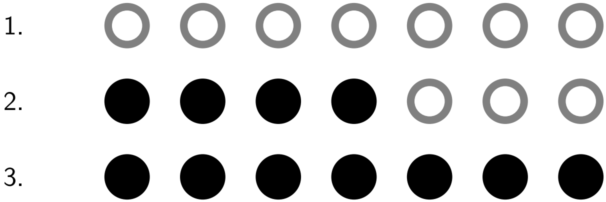

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3.

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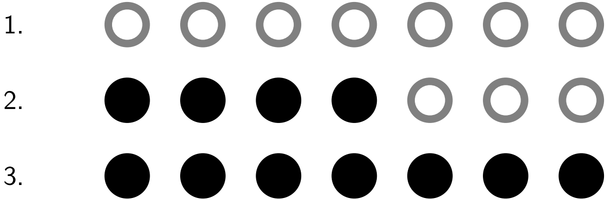
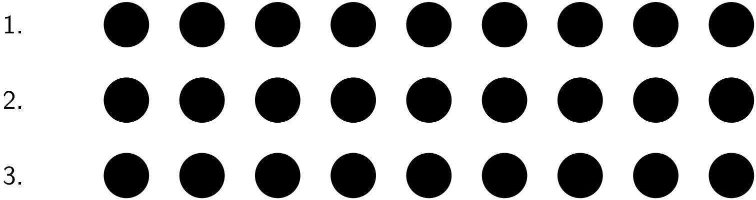
Coded by an octal number $\mathbf{0.d_1 \ldots d_t}$



Example of octal games

- Nim: **0.33333333333333333...**



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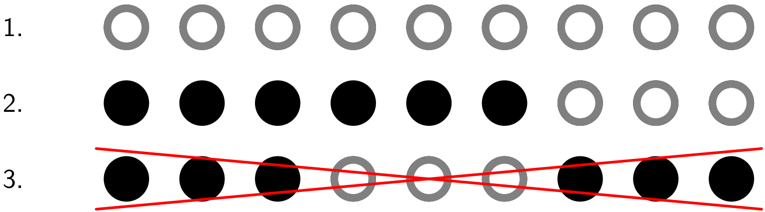

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- Kayles: **0**.**77**.

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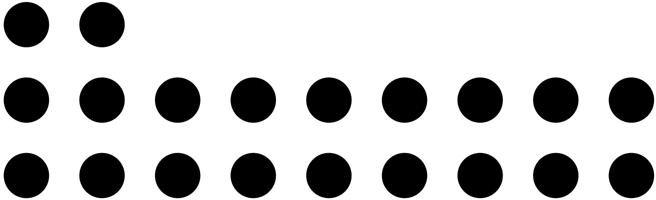

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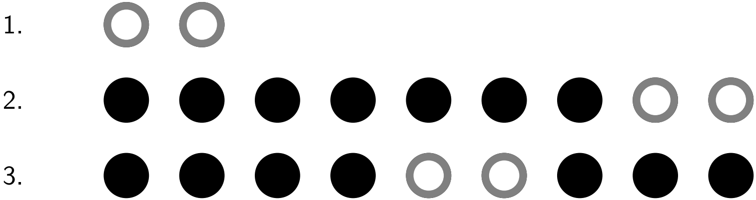

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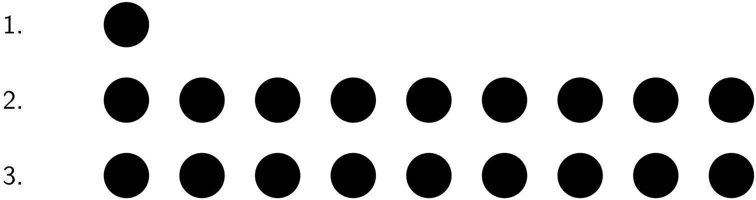

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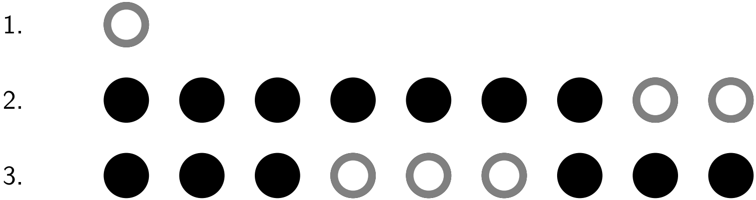

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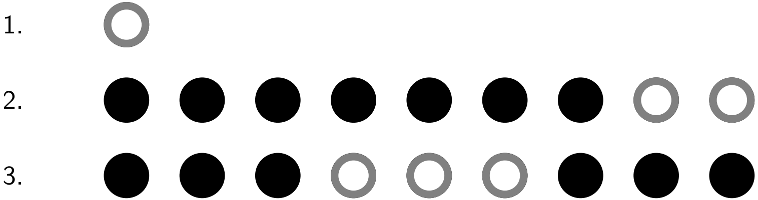

Conjecture (Guy)

Octal games with finite code have a Grundy sequence ultimately periodic.

Hexadecimal games

Why stop at two heaps ?

4.

Hexadecimal games

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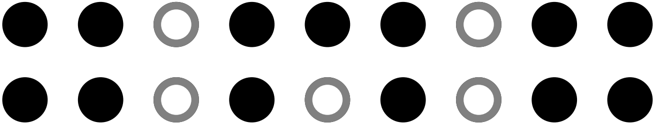
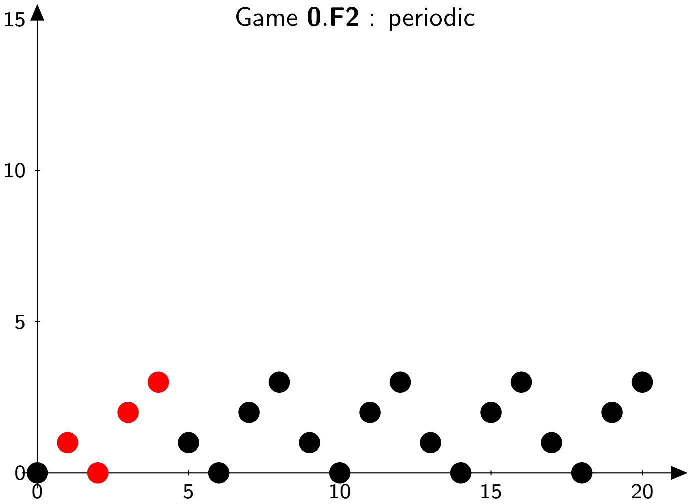
5.



⇒ we can simply generalize to leaving any number of heaps...

Grundy sequences



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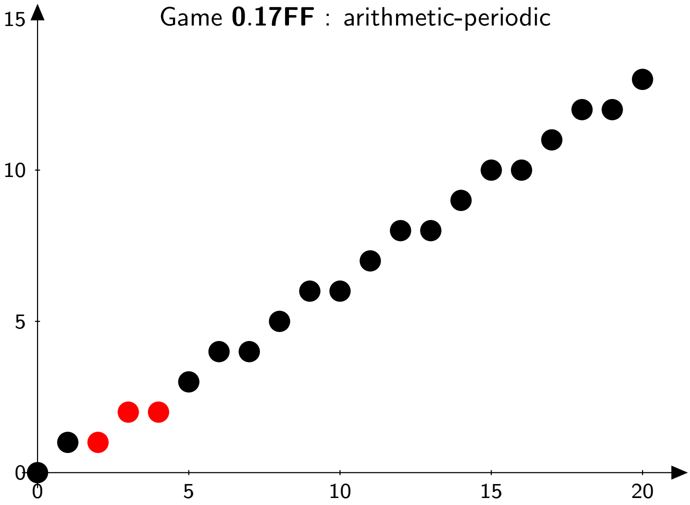

Periodicity and arithmetic-periodicity results

Theorem (WW)

Let H be the hexadecimal game $\mathbf{0.d_1 \ldots d_t}$. *If there exist e and p such that:*

$$\forall e < n \leq 3(e+p)+t,\ \mathcal{G}(n+p) = \mathcal{G}(n)$$

then the Grundy sequence of H is periodic of period p and with pre-period e.

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Theorem (R. Nowakowski)

Let H be the hexadecimal game $\mathbf{0.d_1 \dots d_t}$. If there exist e, $3p \geq t+2$, $s = 2^{\gamma-1} + j$, $j < 2^{\gamma-1}$ such that:

1. $\forall e < n < t + \alpha_{e,\gamma,j} p, \quad \mathcal{G}(n+p) = \mathcal{G}(n) + \mathbf{s}$,
2. $\mathcal{G}([\![0, e]\!]) \subset [\![\mathbf{0}, \mathbf{s}-\mathbf{1}]\!]$ *and* $\mathcal{G}([\![0, e+p]\!]) \subset [\![\mathbf{0}, \mathbf{2s}-\mathbf{1}]\!]$,
3. $\exists d_{2v+1}, d_v \geq 8, \forall \mathbf{g} \in [\![\mathbf{0}, \mathbf{2s}-\mathbf{1}]\!], \exists n > 0, \mathcal{G}(n) = \mathbf{g}$ *or*
   $\exists d_u \geq 8, \forall \mathbf{g} \in [\![\mathbf{0}, \mathbf{2s}-\mathbf{1}]\!], \exists 2v+1, 2w \geq 0, \mathcal{G}(2v+1) = \mathcal{G}(2w) = \mathbf{g}$,

then the Grundy sequence of H is arithmetic-periodic of period p, pre-period e and saltus s.

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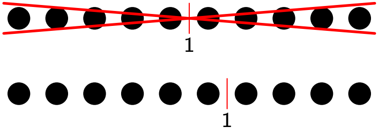
Sommaire

Grundy's Game

A move consists in taking a heap and splitting it into two non-empty heaps of different sizes.
No removing allowed.



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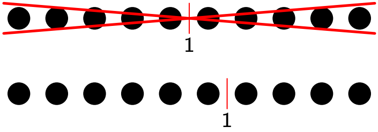


Conjecture (Belekamp, Conway, Guy)

The Grundy sequence of Grundy's game is ultimately periodic.

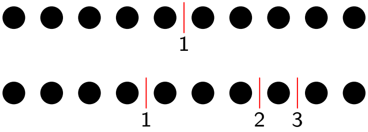
Remark: 2^{35} first values computed (Flammenkamp) without any further clue...

Pre-Grundy Games

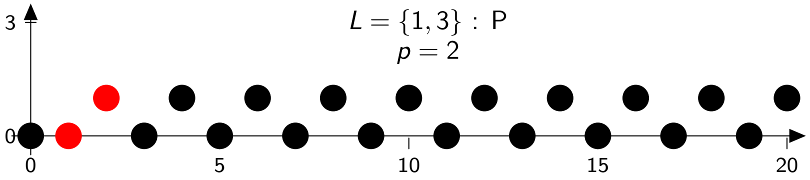
Definition

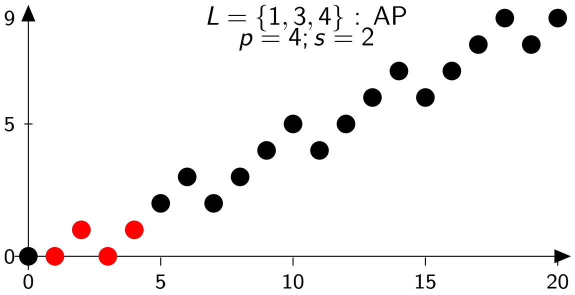
Let $L = \{\ell_1, \ldots, \ell_k\}$ be a list of positive integers. A move on the game $\textsc{PreG}(L)$ consist on splitting a heap of $n \geq \ell_j$ tokens into $\ell_j + 1$ non-empty heaps.

$$L = \{1, 3\}$$



Behavior





Firsts results

$L = \{\ell_1, \ldots, \ell_k\}$	*type*	*Sequence*
$1 \notin L$	*AP*	$(0)^{\ell_1}\ (+1)$

Proofs: For $L = \{\ell_1, \ldots, \ell_k\}$, $\ell_i > 1$.
For $n = a\ell_1 + b + 1$, $b < \ell_1$, let us prove that $\mathcal{G}(n) = a$.
First: $\mathcal{G}(n) \geq a$.

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- ℓ_1 even:

$$O_{\ell_1} = (i\ell_1 + b + 1, a - i, \ldots, a - i)$$

$$\mathcal{G}(O_{\ell_1}) = \mathcal{G}(i\ell_1 + b + 1) = \mathbf{i}$$

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For $n=a\ell_1+b+1$, $b<\ell_1$, let us prove that $\mathcal{G}(n)=a$.
First: $\mathcal{G}(n)\geq a$.

- ℓ_1 odd:

– if $a-i$ odd:

$$O_{\ell_1}=\left(i\ell_1+b+1,1,\frac{a-i-1}{2}\ell_1+1,\frac{a-i-1}{2}\ell_1+1,1,\dots,1\right)$$

$$\mathcal{G}(O_{\ell_1})=\mathcal{G}(i\ell_1+b+1)\oplus\mathcal{G}(1)=\mathbf{i}$$

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– if $a - i$ even:

$$O_{\ell_1} = \left(i\ell_1 + b + 1, 2, \frac{a-i-1}{2}\ell_1 + \frac{1}{2}, \frac{a-i-1}{2}\ell_1 + \frac{1}{2}, 1, \ldots, 1\right)$$

$$\mathcal{G}(O_{\ell_1}) = \mathcal{G}(i\ell_1 + b + 1) \oplus \mathcal{G}(2) = \mathbf{i}$$

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In all cases $\mathcal{G}(O_{\ell_1}) = \mathbf{i}$ for $i < a \Rightarrow \mathcal{G}(n) \geq a$.

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Assume $\mathcal{G}(O_\ell) = a$ for some option of n.

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$$O_\ell = (a_0\ell_1 + b_0 + 1, \ldots, a_\ell\ell_1 + b_\ell + 1)$$

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- $a\ell_1 + b + 1 = \sum a_i\ell_1 + b_i + 1 \quad \Rightarrow \quad \sum a_i \leq a$

$\Rightarrow \sum a_i = a$ and $b \geq \ell \geq \ell_1$... absurd.

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$\{1, \ell_2, \ldots, \ell_k\}(\ell_i \text{ odd })$	P	$(01)^*$
$\{1, 2, 3, \ell_4, \ldots, \ell_k\}$	AP	$(0)^1\ (+1)$
$\{1, 3, 2\ell\}(\ell \geq 2)$	AP	$(01)^\ell\ (+2)$

Main result: arithmetic-periodicity test

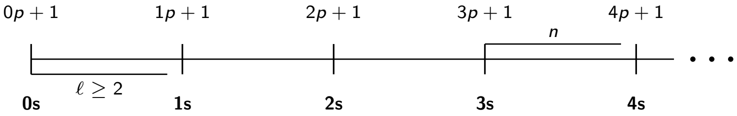
Definition

The game $\mathrm{PreG}(\{\ell_1, \ldots, \ell_k\})$, $\ell_k \geq 2$, satisfies the test *AP* if there exists $p > 0$ and $s = 2^j$ such that:

AP1. For $n \leq 3p$, $\mathcal{G}(n + p) = \mathcal{G}(n) + \mathbf{s}$

AP2. $\mathcal{G}([\![1, p]\!]) = [\![\mathbf{0}, \mathbf{s} - \mathbf{1}]\!]$

AP3. For $n \in [\![3p + 1, 4p]\!]$, $\mathbf{g} \in [\![\mathbf{0}, \mathbf{s} - \mathbf{1}]\!]$, there is an option $O_{\ell,n}$ of n such that $\mathcal{G}(O_{\ell,n}) = \mathbf{g}$ and $\ell \geq 2$.



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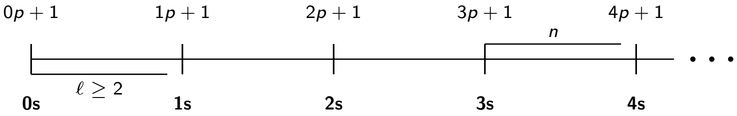

Remarks:

- Sometimes ($AP1$ and $AP2$) $\Rightarrow$ $AP3$.

AP-Theorem

Theorem

Let $L = \{\ell_1, \ldots, \ell_k\}$, $\ell_k \geq 2$, *such that* $\textrm{PreG}(L)$ *verifies the AP-test. Then for all* $n \geq 1$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$.

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Then for all $n \geq 1$*,* $\mathcal{G}(n+p) = \mathcal{G}(n) + s$*.*

Let us show that for $n = ap + 1 + b$:

(A) $\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b)$

(B) for $\mathbf{g} \in [\![\mathbf{0}, (\mathbf{a}-\mathbf{1})\mathbf{s}-\mathbf{1}]\!]$, $\exists O_{\ell,n}$ of n, with $\ell \geq 2$ and $\mathcal{G}(O_{\ell,n}) = \mathbf{g}$.

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Let $L = \{\ell_1, \ldots, \ell_k\}$, $\ell_k \geq 2$, such that $\mathrm{PreG}(L)$ verifies the AP-test. Then for all $n \geq 1$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$.

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(Almost) clear for $n \leq 4$:

Lemma

If $n \leq 4p$, then n verifies $(\mathbf{B})$.

Let $n \geq 4p$.

Proof: $\mathcal{G}(n) \le \mathbf{as} + \mathcal{G}(1+b)$

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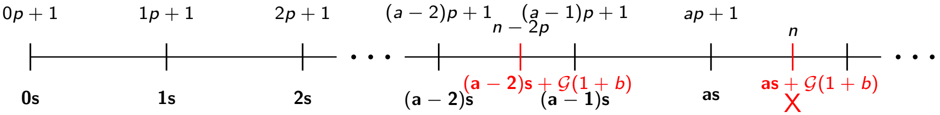
(B) for $\mathbf{g} \in [\![\mathbf{0}, (\mathbf{a}-\mathbf{1})\mathbf{s}-\mathbf{1}]\!]$, $\exists O_{\ell,n}$ of n, with $\ell \ge 2$ and $\mathcal{G}(O_{\ell,n}) = \mathbf{g}$.

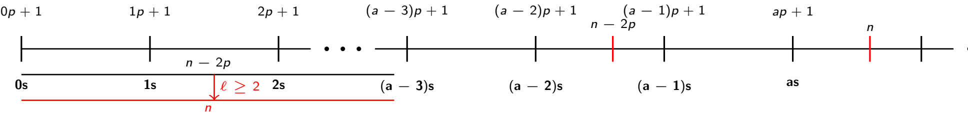


$$\mathcal{G}(O_{\ell,n}) = \mathbf{as} + \mathcal{G}(1+b)$$
$$\Rightarrow \exists O_{\ell,n-2p}, \mathcal{G}(O_{\ell,n-2p}) = \mathcal{G}(n-2p)...$$

Proof: $\mathcal{G}(n) \geq (\mathbf{a}-\mathbf{1})\mathbf{s}$

(A) $\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b)$

(B) for $\mathbf{g} \in [\![\mathbf{0}, (\mathbf{a}-\mathbf{1})\mathbf{s}-\mathbf{1}]\!]$, $\exists O_{\ell,n}$ of n, with $\ell \geq 2$ and $\mathcal{G}(O_{\ell,n}) = \mathbf{g}$.



$$\mathcal{G}(O_{\ell,n-2p}) = \mathbf{g}, \quad \mathbf{g} < (\mathbf{a}-\mathbf{3})\mathbf{s}, \ell \geq 2$$
$$\Rightarrow \exists O_{\ell,n}, \mathcal{G}(O_{\ell,n}) = \mathbf{g}$$

Proof: $\mathcal{G}(n) \geq (\mathbf{a}-\mathbf{1})\mathbf{s}$

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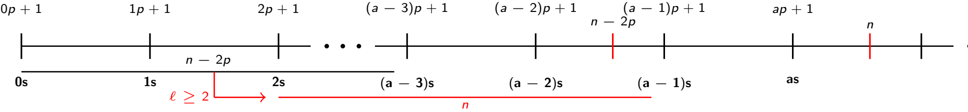

$$\mathcal{G}(O_{\ell,n-2p}) = \mathbf{g}, \quad \mathbf{g} < (\mathbf{a}-\mathbf{3})\mathbf{s}, \ell \geq 2$$
$$\Rightarrow \exists O_{\ell,n}, \mathcal{G}(O_{\ell,n}) = \mathbf{g} + \mathbf{2s}$$

Moreover, n verifies **(B)**.

Proof: $\mathcal{G}(n) \geq \mathbf{as} + \mathcal{G}(1+b)$

(A) $\mathcal{G}(n) = \mathbf{as} + \mathcal{G}(1+b)$
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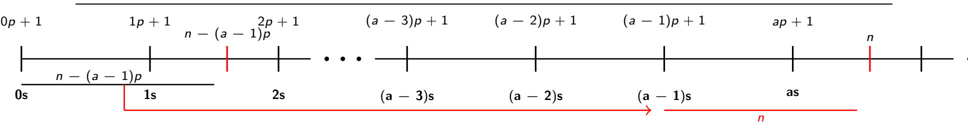

$$\mathcal{G}(O_{\ell,n-(a-1)p}) = \mathbf{g}, \quad \mathbf{g} < \mathbf{s} + \mathcal{G}(\mathbf{1}+\mathbf{b})$$
$$\Rightarrow \exists O_{\ell,n}, \mathcal{G}(O_{\ell,n}) = \mathbf{g} + (\mathbf{a}-\mathbf{1})\mathbf{s}$$

$\Rightarrow$ n verifies also **(A)**

Conjectures and remarks

In summary:

$L = \{\ell_1, \dots, \ell_k\}$	*type*	*Sequence*	*S*
$1 \notin L$	*AP*	$(0)^{\ell_1}\ (+1)$	■
$\{1, \ell_2, \dots, \ell_k\}(\ell_i \text{ odd })$	*P*	$(01)^*$	■
$\{1, 2, 3, \ell_4, \dots, \ell_k\}$	*AP*	$(0)^1\ (+1)$	■
$\{1, 3, 2\ell\}(\ell \geq 2)$	*AP*	$(01)^{\ell}\ (+2)$	■
$\{1, 2\ell\}(\ell \geq 2)$	*AP*	$(01)^{\ell}(23)^{\ell}14(54)^{\ell-1}(32)^{\ell}(45)^{\ell}(67)^{\ell}\ \ (+8)$	C
$\{1, 2\ell_1, 2\ell_2 + 1\}, \ell_i > 1$	*AP*	$(01)^{\ell_1}\ \ (+2)$	C

Conjectures:

1. $\{1, 2\ell\}$, $\ell \geq 2$: OK for $\ell \in \{2, 3, 4\}$ by *AP*-test
2. $\ell_1, \ell_2 > 1$, $\{1, 2\ell_1, 2\ell_2 + 1\}$: OK for $\{1, 4, 5\}$, $\{1, 4, 7\}$, $\{1, 5, 6\}$ by *AP*-test

Conjectures and remarks

In summary:

$L = \{\ell_1, \ldots, \ell_k\}$	*type*	*Sequence*	S
$1 \notin L$	AP	$(0)^{\ell_1}\ (+1)$	■
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$\{1, 2, 3, \ell_4, \ldots, \ell_k\}$	AP	$(0)^1\ (+1)$	■
$\{1, 3, 2\ell\}(\ell \geq 2)$	AP	$(01)^\ell\ (+2)$	■
$\{1, 2\ell\}(\ell \geq 2)$	AP	$(01)^\ell(23)^\ell 14(54)^{\ell-1}(32)^\ell(45)^\ell(67)^\ell\ \ (+8)$	C
$\{1, 2\ell_1, 2\ell_2 + 1\}, \ell_i > 1$	AP	$(01)^{\ell_1}\ \ (+2)$	C
$\{1, 2\}$	???	???	

Conjectures:

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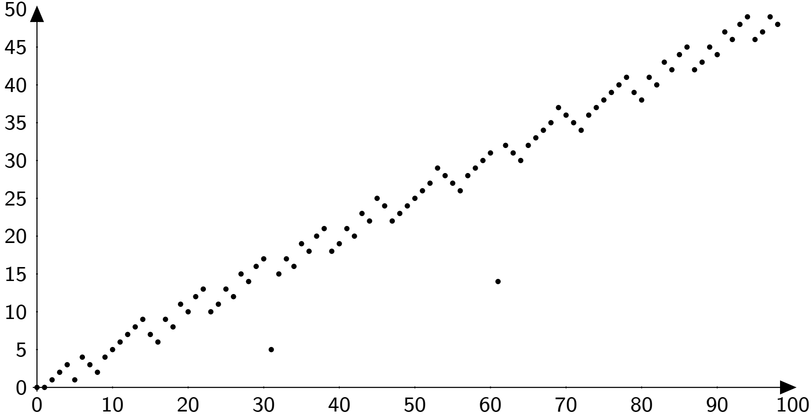
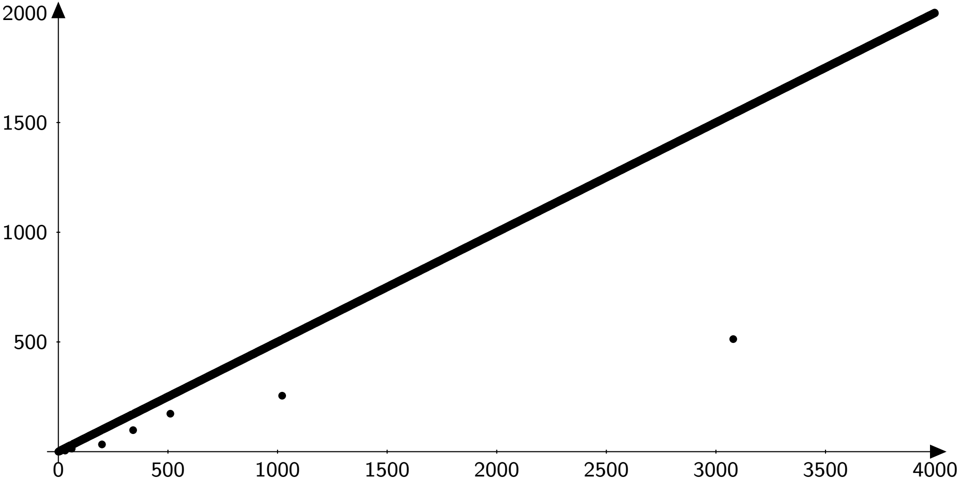


$L = \{1, 2\}$



...maybe not!

Thank you!