Progress and Problems in Restricted Misere Play

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  - Everything is awful!

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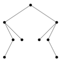
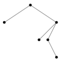
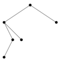
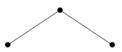
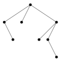
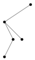
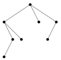
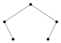
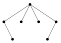
P				
L				
R				
N				

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$\mathcal{R}$	?	$\mathcal{R}$	$\mathcal{R} \cup \mathcal{N}$	$\mathcal{R}$
$\mathcal{N}$	$\mathcal{L} \cup N$	$\mathcal{R} \cup \mathcal{N}$	?	$\mathcal{N}$
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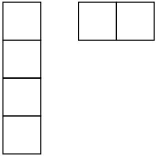

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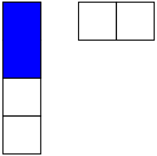

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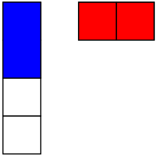

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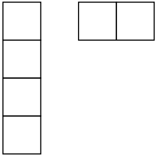

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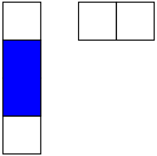

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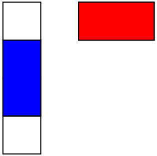

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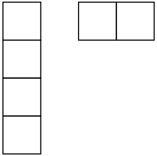

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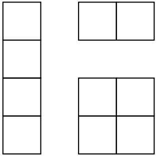

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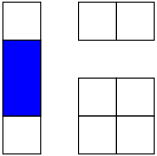

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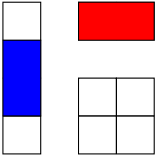

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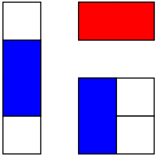

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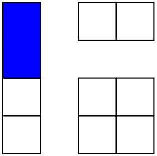

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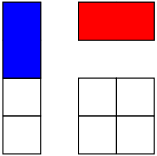

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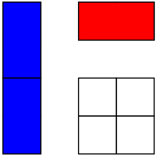

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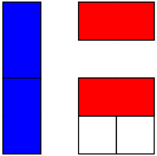

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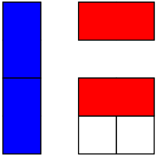
    e.g., 0 and $1 = \{0|\cdot\}$ are incomparable.



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  - A position may have algebraic structure *modulo* $\mathcal{U}$ that it doesn't have in general (e.g., invertibility, simplification).

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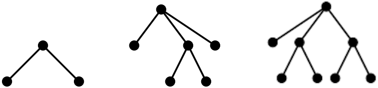

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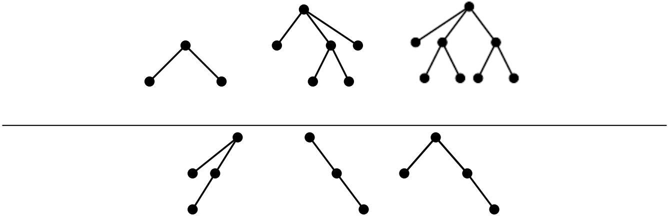

Some definitions and nice universes.

- **Dicot**: at every point, either both players can move or neither can. Let $\mathcal{D}$ be the universe of dicot games.
- **Left end:** Left has no move now.
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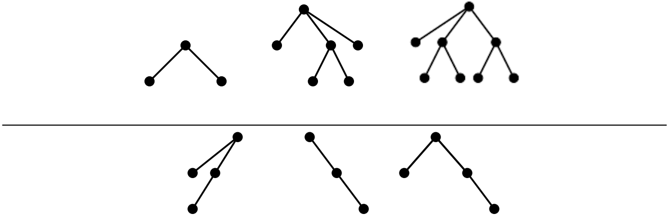

$\hookrightarrow$ Domineering, Hackenbush, NoGo, Snort $\subset \mathcal{E}, \mathcal{D} \subset \mathcal{E}$.

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Let $\mathcal{U}$ *be any universe and let* $S \subset \mathcal{U}$ *be closed under followers. If Left wins playing first on* $G + \overline{G} + Y$ *for all* $G \in S$ *and left ends* $Y \in \mathcal{U}$, *then* $G + \overline{G} \equiv_{\mathcal{U}} 0$.

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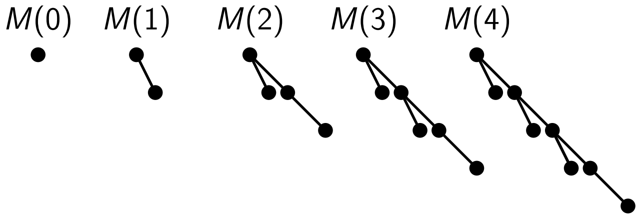

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Future directions

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- Apply results for dead-ending universe to specific rule sets within $\mathcal{E}$, such as domineering, in order to solve such games under misère play.

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