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Computing the Shapley value of graph games with restricted coalitions

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1. **Introduction**
   - Classical cooperative games
   - Restricted cooperation
2. **Graph games on a product of chains**

Outline

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2. **Graph games on a product of chains**

Introduction

A cooperative game is a pair (N, v) where

i) N is a finite set of players.

ii) $v : \; 2^N \rightarrow \mathbb{R}$ is a function with $v(\emptyset) = 0$.

Question

How the players will share the value $v(N)$?

An answer: the Shapley value [L.S. Shapley 1953]

$$\varphi_i = \sum_{S \ni i} \frac{(|S|-1)! \cdot (n-|S|)!}{n!} [v(S) - v(S \setminus \{i\})]$$

The vector φ is called the Shapley value of the game (N, v).

The Shapley value was obtained by imposing a set of axioms that the solution must satisfy: efficiency, linearity, symmetry, null player.

Restricted cooperation

A problem

I practice not all coalitions are feasible: language barriers, geography, hierarchies.

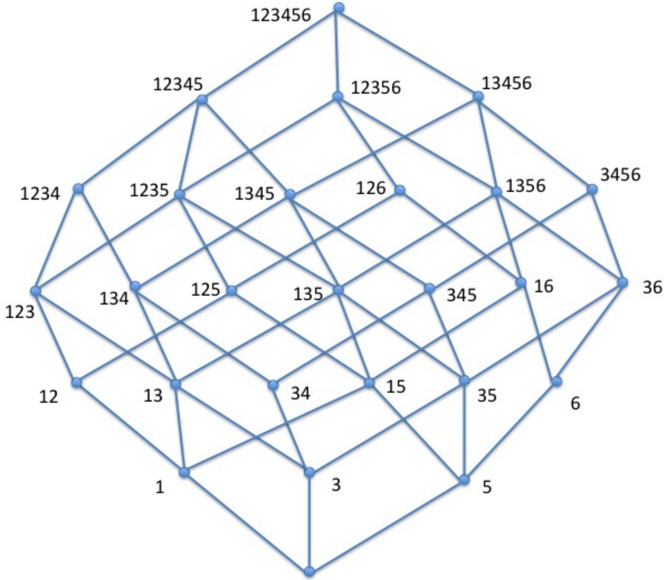
implicational systems

Let $\Sigma = \{A_1 \rightarrow a_1, ..., A_m \rightarrow a_m\}$ be an implicational system on N and $X \subseteq N$. The Σ-closure of X, denoted X^Σ, is the smallest set containing X and satisfying: $\forall 1 \leq j \leq m, A_j \subseteq X^\Sigma \Rightarrow a_j \in X^\Sigma$.

The set $\mathcal{F}_\Sigma = \{X^\Sigma, X \subseteq N\}$ is a closure system (closed under intersection and containing N) and hence is a lattice (a partially ordered set where any two elements have a least upper bound and a greatest lower bound).

Example

$$\Sigma = \{2 \rightarrow 1, 4 \rightarrow 3, 6 \rightarrow 5\}$$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Generalization of the Shapley value [Faigle et al 2016]

For a maximal chain c and $i \in N$, we denote by $F(c, i)$ the last coalition in c that doesn't contain the player i, and by $F^+(c, i)$ the first coalition in c that contains the player i.

$$\varphi_i(v) = \frac{1}{|Ch|} \sum_{c \in Ch} \frac{v(F^+(c,i)) - v(F(c,i))}{|F^+(c,i) \setminus F(c,i)|}. \quad (1)$$

Define the set

$$\mathcal{A}_i = \{(F, F') \in \mathcal{F}_\Sigma^2 \mid \exists c \in Ch : F = F(c,i) \text{ and } F' = F^+(c,i)\}\,.$$

For any $F \in \mathcal{F}_\Sigma$, we denote by $Ch^{\downarrow}(F)$ (resp. $Ch^{\uparrow}(F)$) the number of maximal chains of the sublattice $[\emptyset, F]$ (resp. $[F, N]$). With this notation, equation (1) becomes

$$\varphi_i(v) = \frac{1}{Ch^{\downarrow}(N)} \sum_{(F,F') \in \mathcal{A}_i} \frac{Ch^{\downarrow}(F) \cdot Ch^{\uparrow}(F')}{|F' \setminus F|}(v(F') - v(F))\,. \qquad (2)$$

Outline

We have a partial order $(P, \preccurlyeq)$ on N, which is the disjoint union de chains of the same length.

$$i \rightarrow j \in \Sigma \Leftrightarrow i \preccurlyeq j$$

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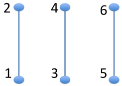

$\mathcal{F}_\Sigma$ is isomorphic to the product of the chains of the order $(P, \preccurlyeq)$.

Graph games

The model of weighted graph games captures the interactions between pairs of players. This is done by considering an undirected graph $G = (N, E)$ with an integer weight v_{ij} for each edge $\{i, j\} \in E$.
We define a cooperative game (N, Σ, v) by:

$$v(S) = \sum_{\{i,j\} \subseteq S} v_{ij} \quad \forall S \in \mathcal{F}_\Sigma.$$

Idea

Partition $\mathcal{A}_i$ in such a way that $Ch^{\downarrow}(F) \cdot Ch^{\uparrow}(F^{+})$ is constant inside each block of the partition.

Proposition 3

let $i \in N$ and $c(i)$ the chain containing i in P. The elements $\mathcal{A}_i$ are exactly the pairs $(F \cup \{i\}^{\Sigma} \setminus \{i\}, F \cup \{i\}^{\Sigma})$ where $F \in \mathcal{F}_{\Sigma}$ with $F \cap c(i) = \emptyset$.

The set $\mathcal{A}_i$ can thus be identified with

$$\tilde{\mathcal{A}}_i = \{F \in \mathcal{F}_{\Sigma} \mid F \cap c(i) = \emptyset\}.$$

The partition

We define an equivalence relation $\mathcal{R}_i$ over $\tilde{\mathcal{A}}_i$ as follows:

$$F_1 \mathcal{R}_i F_2 \Leftrightarrow P_{|F_1} \text{is isomorphic to} P_{|F_2}.$$

Encoding the equivalence classes

The next proposition gives an encoding of the class $\overline{F}$, with $|F| = k$, by a vector of integers in the set:

$$\mathcal{D}_k = \{(x_0, \ldots, x_l) \in \mathbb{N}^{l+1}, \text{ such that } \sum_{t=0}^{l} x_t = m - 1, \sum_{t=0}^{l} t \cdot x_t = k\}.$$

proposition 4

Let $i \in N$. The sets $\mathcal{Q}_i$ and $\mathcal{E} = \bigcup_{k=0}^{n-l} \mathcal{D}_k$ are in bijection by the mapping $\psi : \mathcal{Q}_i \rightarrow \mathcal{E}$, $\overline{F} \mapsto \psi(\overline{F}) = (x_0, \ldots, x_l)$ where x_t is the number of chains of size t in $P_{|F}$ for $1 \leq t \leq l$, and $x_0 = m - 1 - \sum_{t=1}^{l} x_t$.

Furthermore, we have $\psi(\overline{F}) \in \mathcal{D}_k$ with $k = |F|$.

Proposition 5

We have $|\mathcal{D}_k| \in O(k^l)$.

Notation

Let $x \in \mathcal{E}$ an denote by $\mathcal{A}_i^x$ the class $\psi^{-1}(x)$

Lemma 1

Assume that all the chains of P have the same length and let $x \in \mathcal{E}$. Then for all $F_1, F_2 \in \mathcal{A}_i^x$, we have:

$$Ch^{\downarrow}(F_1 \cup \{i\}^{\Sigma} \setminus \{i\}) \cdot Ch^{\uparrow}(F_1 \cup \{i\}^{\Sigma}) = Ch^{\downarrow}(F_2 \cup \{i\}^{\Sigma} \setminus \{i\}) \cdot Ch^{\uparrow}(F_2 \cup \{i\}^{\Sigma}).$$

Notation

Pour $F \in \mathcal{A}_i^x$:

$$\alpha_x = Ch^{\downarrow}(F \cup \{i\}^{\Sigma} \setminus \{i\}) \cdot Ch^{\uparrow}(F \cup \{i\}^{\Sigma})$$

Lemma 2

Let $x \in \mathcal{E}$ and $k = \sum_{t=0}^{l} t \cdot x_t$. We have

$$\alpha_x = \frac{(k + h(i))! \cdot (n - k - h(i) - 1)!}{h(i)! \cdot (l - h(i) - 1)! \cdot \prod_{t=0}^{l} [t! \cdot (l - t)!]^{x_t}}$$

Proposition 6

Let (N, Σ, v) be a weighted graph game and $i \in N$. We have,

$$\varphi_i(v) = \frac{1}{Ch^{\downarrow}(N)} \sum_{k=0}^{n-l} \sum_{x \in \mathcal{D}_k} \sum_{j \neq i} \beta_{ij}^{x} \cdot \alpha_x \cdot v_{ij}, \text{ where } \beta_{ij}^{x} = |\{F \in \mathcal{A}_i^{x} \mid j \in F \cup \{i\}^{\Sigma}\}|.$$

Lemma 3

Let $i \neq j \in N$ and $x \in \mathcal{E}$. Then

$$\beta_{ij}^{x} = \begin{cases} 0, \quad si \;\; j \to i, \\[2em] \dfrac{(m-1)!}{\prod_{t=0}^{l} x_t!}, \quad si \;\; i \to j, \\[3em] \dfrac{(m-2)!}{\prod_{t=0}^{l} x_t!} \cdot \displaystyle\sum_{t=h(j)+1}^{l} x_t, \quad \text{sinon.} \end{cases}$$

Theorem 1

The Shapley value φ_i of a player i in a weighted graph game on a product of m chains with the same length $l - 1$ can be computed in $O(n^{l+3})$, where n is the number of players. For fixed l, it can be computed in polynomial time.

Thank you for your attention