Playful game comparison and Absolute CGT

Urban Larsson, Technion - Israel Institute of Technology,
coauthors Richard N. Nowakowski and Carlos P. dos Santos

GAG-2017, Lyon 1

Thanks to organizers

- We develop a framework for many classes (universes) of combinatorial games:

- normal play, misere play, scoring play possibly with restrictions on the games: dicot, dead ending, guaranteed scores, etc

- Similar techniques have been developed by Siegel, Renault, Milley, Ettinger, Stewart, Santos, Nowakowski, Larsson, Dorbec, Sopena et al.

- Since methods are similar for these play conventions, we wish to unify theory

Game comparison

- Basic setting: no chance, 2 players Left and Right, alternating perfect play, a given winning condition, disjunctive sum, etc
- Given two games G and H, in any situation, would you prefer G before H?
- Here “in any situation” means in a disjunctive sum with any game in the same *universe*

- Berlekamp, Conway, Guy: normal play is a group structure and game comparison simplifies to *play* G-H
- G ≥ H if and only if Left wins G - H when Right starts
- Normal play game comparison is *constructive,* a finite computation
- We extend constructive game comparison to other winning conventions
- For each convention, the *free space of games* is defined recursively, starting with each *adorned* empty set of options

Empty sets and their adorns

- Each empty set of options has an *adorn*
- For each game convention, the set of adorns is a group with a neutral element, ‘0’
- In misere and normal play, the set of adorns is {0}
- In scoring play the set of adorns is the set of real numbers

- A game is *atomic*, if at least one player has no options,
- left-atomic if Left has no options; right-atomic if Right has no options
- It is *purely atomic* if both left- and right-atomic

Unifying terminology for 2-player combinatorial games

- First: unify definition of *outcomes* of games
- Normal play and misere play are *last-move* conventions: the outcome depends on who moves last
- For last-move conventions we can use a binary result, say -1 or +1, where Left prefers positive
- A problem to solve: what happens in a disjunctive sum of games?

- In the game G + H, then if G ends, we do not want to assign a binary result to G

- The disjunctive sum ends when both games have ended

- Solution: in last-move conventions, assign a 0 to each terminal situation

- The *evaluation* of an empty set of option in say G is postponed until G+H ends

- In normal play, the situation ‘Left cannot move’ evaluates to -1
- $v(0) = -1$
- In misere play, the situation ‘Left cannot move’ evaluates to +1
- $v(0) = +1$
- For scoring play, $v(a) = a$, if Left (or Right) cannot move evaluates to a

Unified computation of outcomes

- The *outcome* of a game is an ordered pair of results o(G) = (oL(G), oR(G)), where
- oL(G) = v(a) if G is left-atomic with adorn a
- oL(G) = max{oR(GL)} otherwise, where max runs over the left options of G
- oR(G) = v(a) if G is right-atomic with adorn a
- oL(G) = max{oL(GR)} otherwise

Absolute universes

- A set of games is a *universe* if it is closed under taking options, conjugate, and disjunctive sum
- A universe of combinatorial games is *absolute* if it is *parental* and *dense*
- *Parental* means that if $\mathcal{G}$ and $\mathcal{H}$ are sets of games, then the game $\{\mathcal{G}|\mathcal{H}\}$ is also in the universe
- *Dense* means that, for any outcome x, for any game G, then there is a game H such that the o(G+H) = x

The result

- For absolute universes of combinatorial games, game comparison is ‘constructive’; we use a normal play analogy:
- For any games G, H in an absolute universe
- A *dual* normal play game [G, H], also called *Left’s provisonal game* (LPG), is played as follows
- The *Right options* are of the form [GR, H] or [G, HL]

Left must maintain a 'proviso'

- The *Left options* are of the form [GL, H]
- provided that o(GL+X) ≥ o(H+X), for all left-atomic games X
- or [G,HR]
- provided that o(G+X) ≥ o(HR+X), for all right-atomic games X

and a common normal part

- Main Theorem: *For any games G and H in* ***any*** *absolute universe, G ≥ H if and only if Left wins* [*G*,*H*] *in* ***normal play*** *(!) playing second*

- Proof uses *common normal part*: for all GR there is GRL such that GRL ≥ H, or there is HR such that GR ≥ HR

- for all HL there is GL such that GL ≥ HL, or there is HLR such that G ≥ HLR

- The proof of common normal part, given *G ≥ H,* uses the *downlinked idea* developed by Ettinger and Siegel

Downlinked idea for absolute universes

- A game G *downlinks* the game H if there exists a game T such that oL(G+T) < oR(H+T)
- Lemma 1: G ≥ H implies G downlinks no HL and no GR downlinks H (easy)
- Lemma 2: G downlinks H iff for all GL, GL **not** ≥ H and for all HR, G **not** ≥ HR (hard, uses dense and parental)

Simplification

- In a dicot universe, either no player has an option or both players have an option
- Left's proviso simplifies to: $o(G) \geq o(H)$
- Hence game comparison is constructive
- For other absolute universes (guaranteed scoring, Dead ending misere, etc) game comparison is also constructive: see Richard's and Rebecca's talks

Example: Dicot Misere

We adopt notation from Normal-play canonical forms for (Dicot) Misère-play games; for example, $* = \langle 0 \,|\, 0 \rangle$, $\uparrow = \langle 0 \,|\, * \rangle$, $\downarrow = \langle * \,|\, 0 \rangle$. Moreover, we introduce some new symbols $\curlywedgeuparrow = \langle 0, * \,|\, * \rangle$ (pronounced "mup", which means 'misère up'), $\curlyveedownarrow = \langle * \,|\, *, 0 \rangle$ ("mown" = 'Misère down'), $\curlywedgeuparrow * = \langle 0, * \,|\, 0 \rangle$, and $\curlyveedownarrow * = \langle 0 \,|\, *, 0 \rangle$. (Note, in Normal-play, $\curlywedgeuparrow$ and $\curlyveedownarrow$ reduce to simpler games but this is not true in Misère play.)

Take $\mathbb{U}$ as the Dicot misère Universe and let $G = \curlywedgeuparrow = \langle 0, * \,|\, * \rangle$ and $H = 0$. In $[\curlywedgeuparrow, 0]$, Left cannot move to $[*, 0]$, because $P = o(*) \not\geq_{\mathbb{U}} o(0) = N$ gives that the Proviso is not satisfied. The game tree of $[\curlywedgeuparrow, 0]$ is given in Figure 1. This shows that $[G, H] = \uparrow > 0$ and thus $G \succcurlyeq H$.

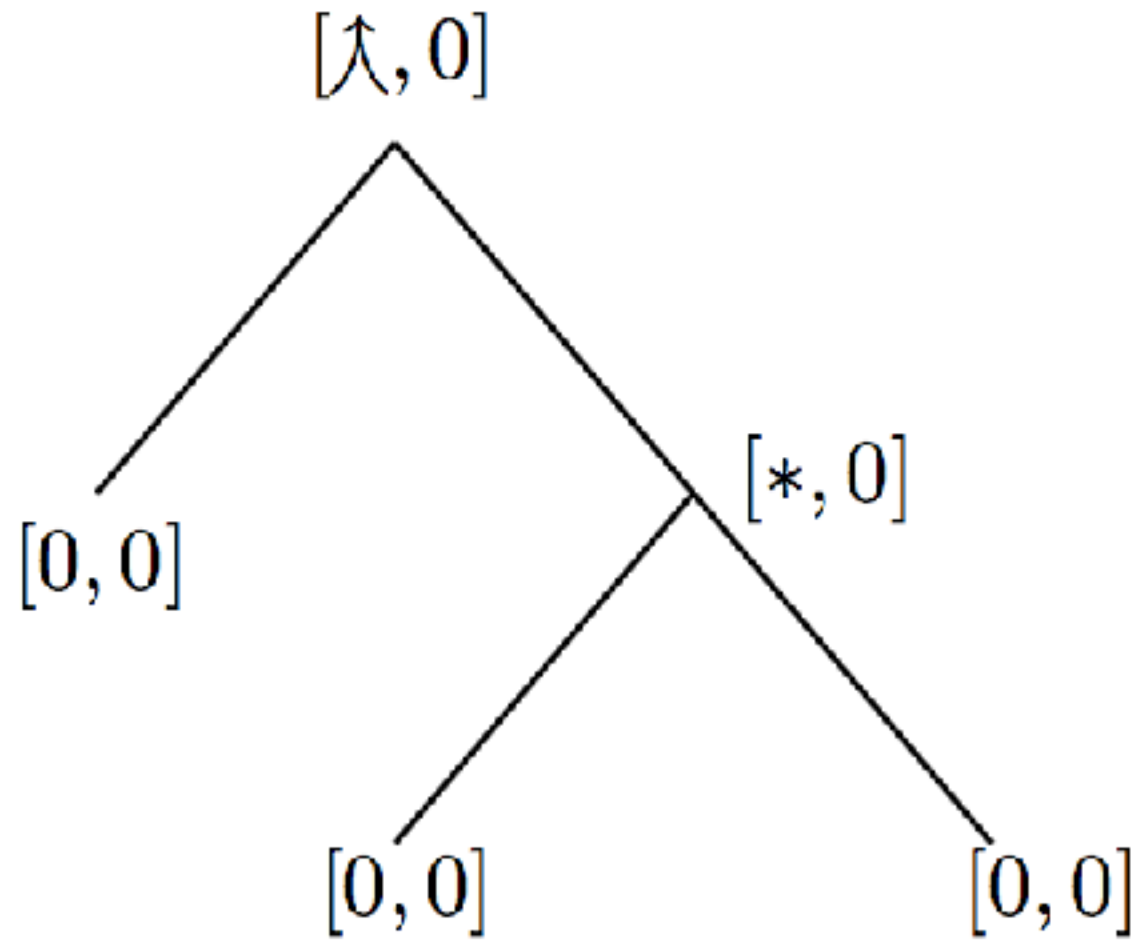

FIGURE 1. The game tree of a dual Normal-play game, the LPG $[\curlywedge, 0]$, with canonical forms $\curlywedge$ and 0 in the Dicot Misère Universe. The edges are the move options from all followers, for Left (left slant) and Right (right slant) respectively.

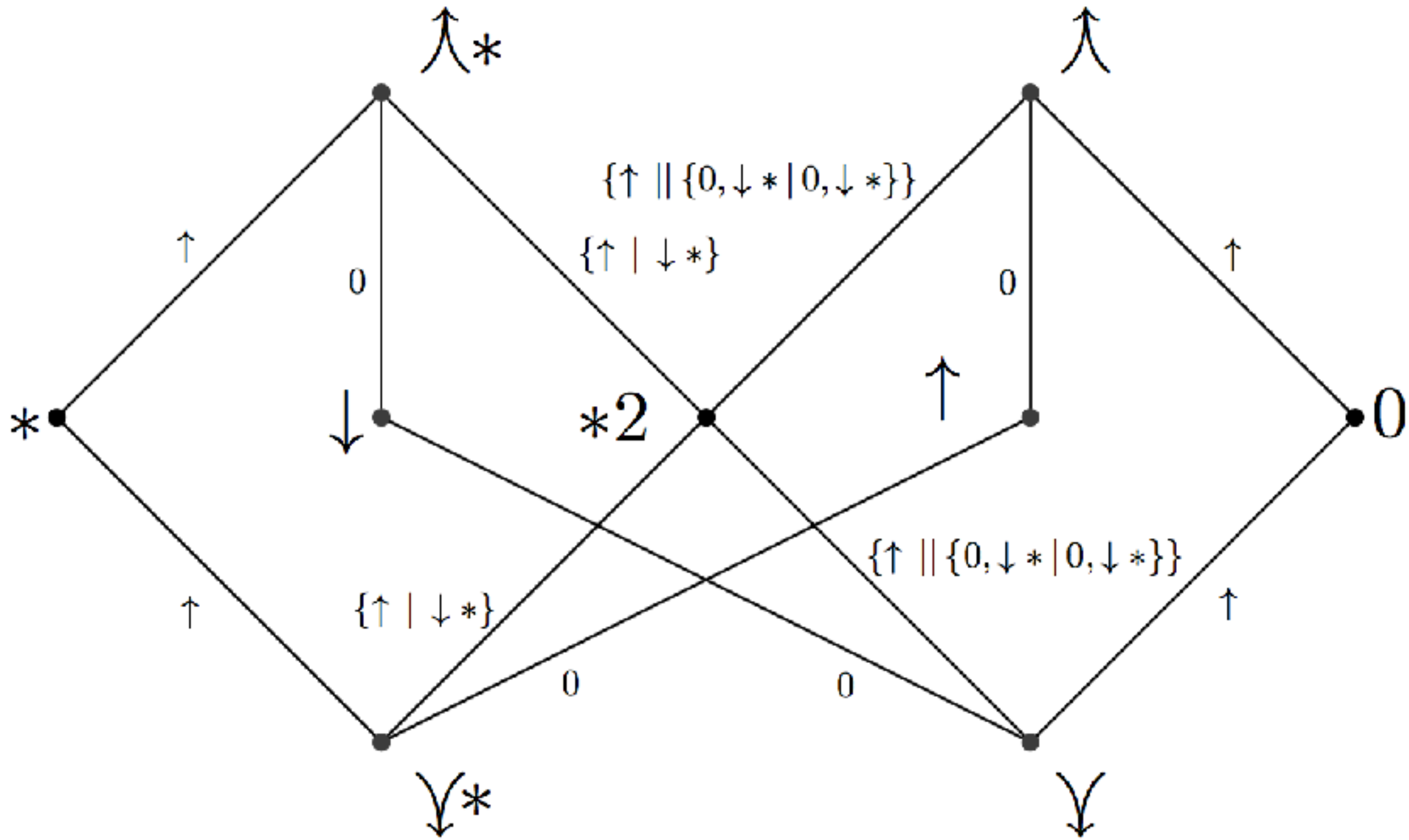

FIGURE 2. The diagram illustrates the partial order of all games (the nodes) of rank 2 in Dicot Misère-play. The edges correspond to the dual Normal-play LPGs $[G, H]$, where the respective 'upper' game is G and the 'lower' is H. We see, for example, that ⋏ is the simplest Dicotic game strictly larger than zero.

Open problems

- To publish the 2 manuscripts. (The first one, which contains all the good ideas got rejected twice. It is probably the strongest paper I wrote.)

- The second manuscript shows that LPG is a category for any absolute universe. It seems that guaranteed scoring play could have interesting categorical structures. Similar to normal play it satisfies a certain closure property. (Dicot absolute universes do not satisfy closure properties.)

- Study some of the infinitely many absolute dicot misere extensions (they are between dicot and dead ending).