Nordhaus-Gaddum inequalities for coloring games

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Joint work with
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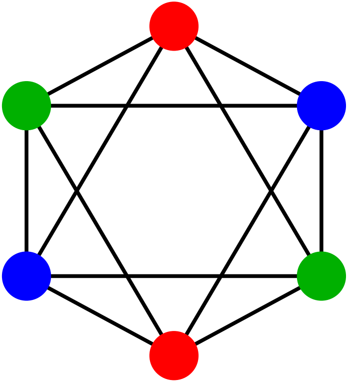
Definitions

Proper coloring

A **coloring** of a graph is the assignment of a color to each vertex of the graph. A coloring is **proper** if two adjacent vertices have different colors. The *chromatic number* of a graph G is denoted by $\chi(G)$.



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Coloring game

The **coloring game** was introduced by Brahms in 1981 and rediscovered in 1991 by Bodlaender.

- At start : a graph G uncolored and a set Φ of colors.
- Alice and Bob take turns coloring an uncolored vertex of G with a color of Φ.
- Alice wins when the graph is fully colored. Bob wins if he can prevent Alice's victory.

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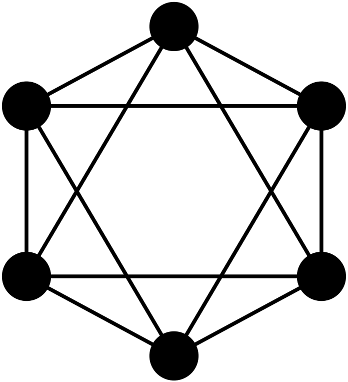
Example

Coloring game

A little game...



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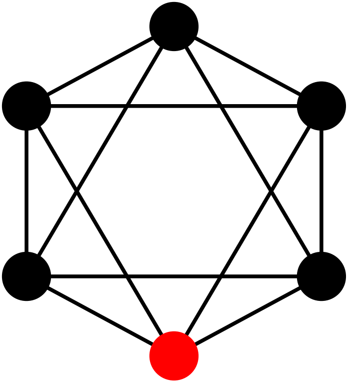

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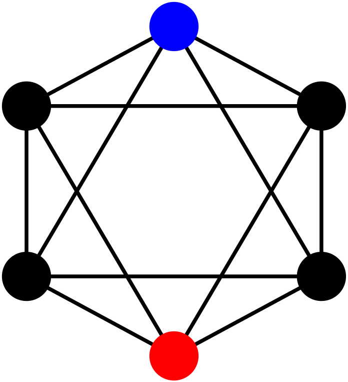

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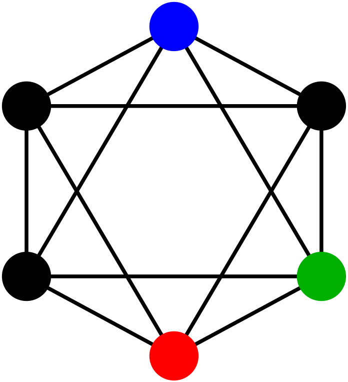

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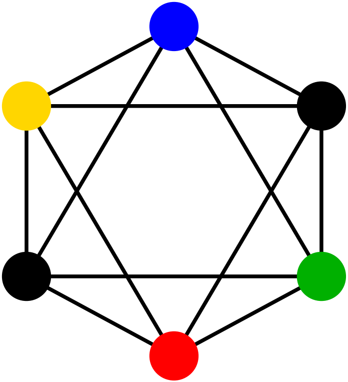

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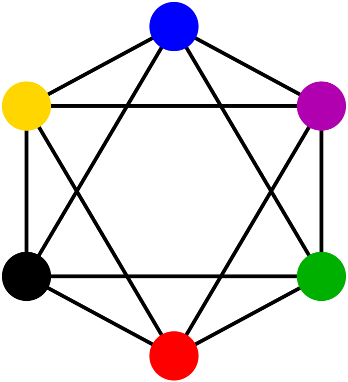

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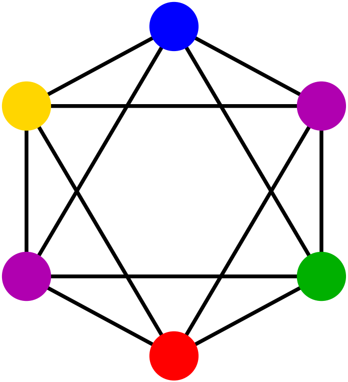


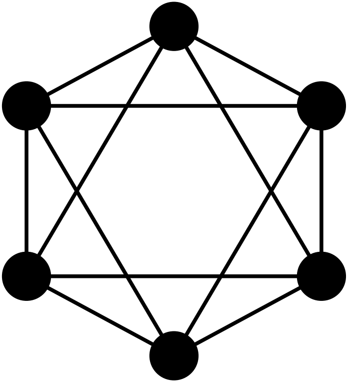
Monotony by subset ?

Question

If Alice has a winning strategy for k colors on a graph G, does she have a winning strategy for k colors on any subgraph of G ?

Coloring game

Another game ?



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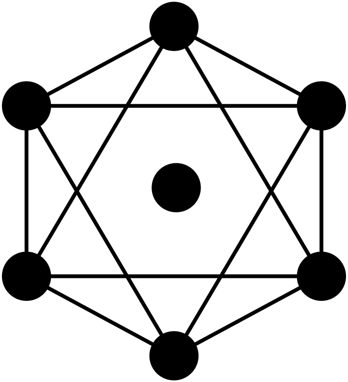

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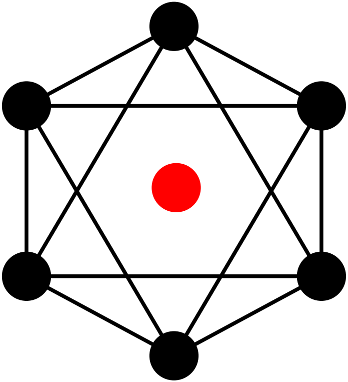

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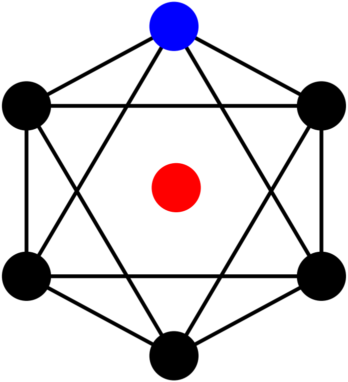

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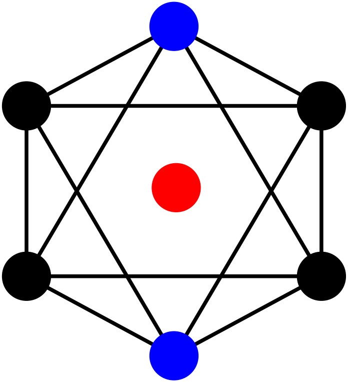

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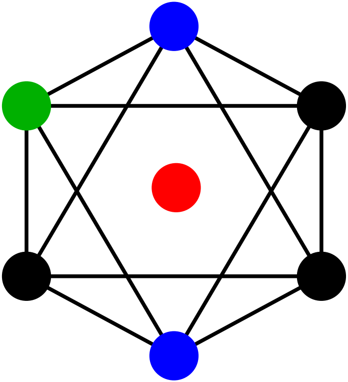

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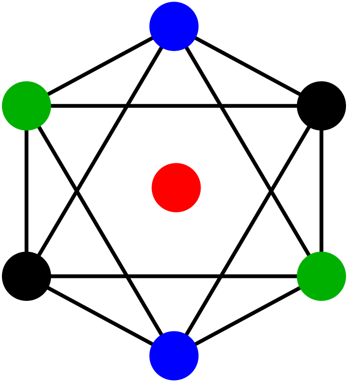

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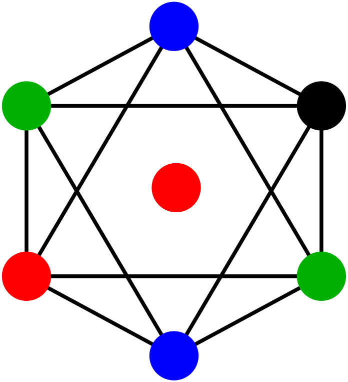

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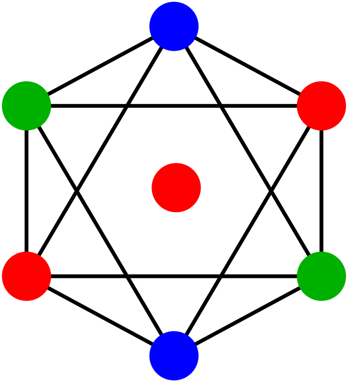

Monotony by number of colors ?

An open problem

Let G be a graph on which Alice has a winning strategy with k colors.

Let $k' > k$.

Does Alice have a winning strategy for G with k' colors ?

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Game chromatic number

Coloring game

The **game chromatic number** of a graph G is the smaller number of colors for which Alice has a winning strategy for G.

Trivial bounds

For any graph G, $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$.

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Norhaus-Gaddum inequalities

Theorem [Nordhaus and Gaddum, 1956]

For any graph G of order n, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

Survey: [Aouiche and Hansen, 2013].

Norhaus-Gaddum inequalities

Result (1)

Theorem [Nordhaus and Gaddum, 1956]

For any graph G of order n, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.
These bounds are tight for an infinite number of values of n.

Theorem

- For any graph G of order n, $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.
- These bounds are asymptotically tight.

For infinite values of n, there are graphs G such that

- $\chi_g(G) + \chi_g(\overline{G}) = 2\sqrt{2n} - 1$ and
- $\chi_g(G) + \chi_g(\overline{G}) = \lceil \frac{4n}{3} \rceil - 1$.

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This lemma is tight for $G = P_1$.
Consider G is a complete $(\sqrt{n})$-partite graph.

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For any graph G of order n, $\chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.

Proof : Assume n is even.
Alice colors with priority vertices with degree at least $\frac{n}{2}$.
Let $A(G)$ be the set of vertices of degree larger than $\frac{n}{2}$,
$B(G) = V(G) - A(G)$.

- $A(G) = B(\overline{G})$ and $B(G) = A(\overline{G})$.
- If $|B(G)| < \lceil \frac{n}{4} \rceil$, then $\chi_g(G) \leq \frac{n}{2}$.
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For any graph G of order n, $\chi_g(G) + \chi_g(\overline{G}) \leq \left\lceil \frac{3n}{2} \right\rceil$.

This lemma is tight for $G = P_1$ and $G = P_4$.
Consider the joint graph $G_l = S_l + K_{\lceil \frac{l}{2} \rceil}$, $n = l + \lceil \frac{l}{2} \rceil$

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Norhaus-Gaddum inequalities

Result (1)

Theorem [Nordhaus and Gaddum, 1956]

For any graph G of order n, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.
These bounds are tight for an infinite number of values of n.

Theorem

- For any graph G of order n, $2\sqrt{n} \leq \chi_g(G) + \chi_g(\overline{G}) \leq \lceil \frac{3n}{2} \rceil$.
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For infinite values of n, there are graphs G such that

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Marking game

Or "colorblind coloring game"

Marking game [Zhu, 1999]

- G is a graph, k an integer.
- Alice and Bob take turns marking an unmarked vertex of G.
- At each turn, the marked vertex must have no more than $k-1$ marked neighbors.
- Alice wins when all the vertices are marked, Bob wins otherwise.
- The smaller k for which Alice has a winning strategy on G is the **coloring game number**, denoted by $col_g(G)$.

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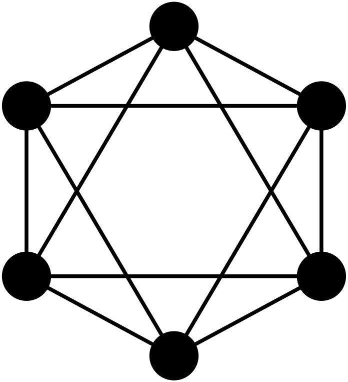
A much easier game to study...



Norhaus-Gaddum inequalities

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Theorem

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For any graph G of order n, $2\lceil \frac{n}{2} \rceil \leq col_g(G) + col_g(\overline{G}) \leq \lfloor \frac{8n-2}{5} \rfloor$.
The lower bound is tight for infinitely many values of n.
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Lemma

For any graph G of order n, $2\left\lceil \frac{n}{2} \right\rceil \leq col_g(G) + col_g(\overline{G})$.

- If n is even, $col_g(G) + col_g(\overline{G}) \leq n$.
- If n is odd, $col_g(G) + col_g(\overline{G}) \leq n + 1$.

Consider the case n is odd.

Proof (sketch) : Order the vertices by increasing degree. Bob always selects the unselected vertex with largest degree.

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- If n is odd, $col_g(G) + col_g(\overline{G}) \leq n + 1$.

When n is odd, this bound is reached by K_n.
When n is even, this bound is reached for every $n \neq 2, 4$.

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Consider the joint graphs $G_l = S_l + K_{l+1}$ and $G_{l'} = S_l + K_{l+2}$.

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Open problems

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Merci !