Computational Complexity of Games

Kyle Burke

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- CS: Fastest algorithm to answer this for entire ruleset?

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  - $10n^2 + 37.4n + 124 \rightarrow 10n^2 \rightarrow n^2$
  - $O(101n^2) = O(n^2)$
  - $O(n^3 + n^2 + n + 50) = O(n^3)$

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Note: No fixed-size rulesets!

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  - $O(n \log(m))$ steps.

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... are $\mathrm{EXPTIME}$-complete.

EXPTIME-complete Rulesets?

Yes, there are rulesets that require exponential time to solve!

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How do we know there's no faster algorithm?

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How does this show hardness?

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  - Assume $\textsc{Banane}$ solvable in faster-than-exponential time by some algorithm A
  - New $\textsc{Chess}$-solving algorithm, $B(x)$: return $A(f(x))$
  - B solves $\textsc{Chess}$!
  - B solves $\textsc{Chess}$ in faster-than-exponential time!
  - Now $\textsc{Chess}$ is not $\mathrm{EXPTIME}$-hard $\rightarrow\leftarrow$

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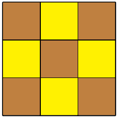

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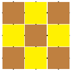

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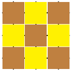

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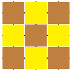

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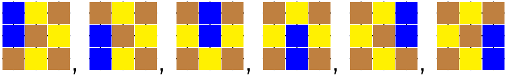

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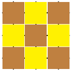

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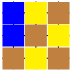
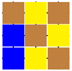
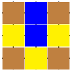
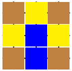
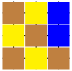
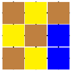
, , , , , 

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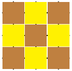

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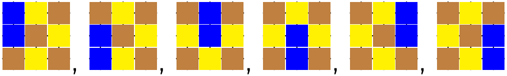

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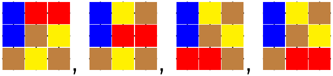

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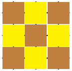

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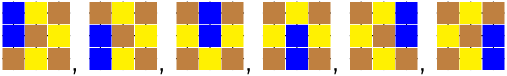

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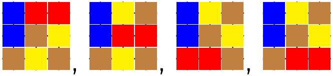

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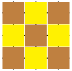

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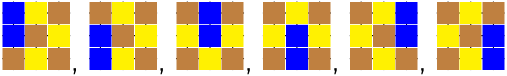

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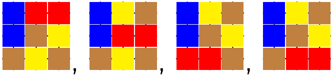

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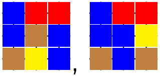

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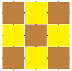

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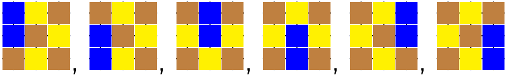

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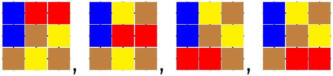

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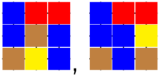

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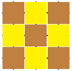

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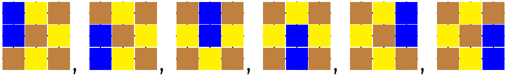

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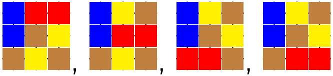

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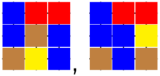

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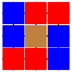 $\in \mathcal{R} \cup \mathcal{P}$ (Right wins)

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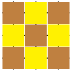

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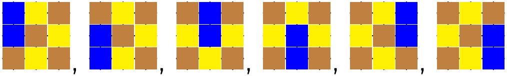

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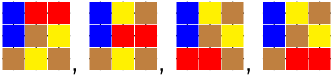

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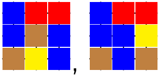
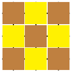


↓ Right's potentially-winning options ✓

So... previous move not a winner for Left. Back up and try the next one.

What about Short Games?



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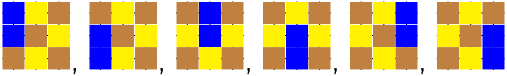

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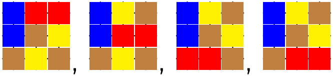

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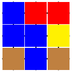

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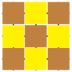

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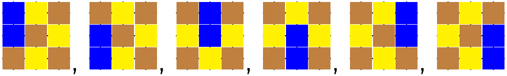

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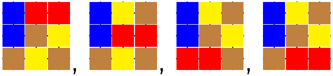

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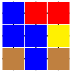
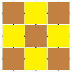


No Right move!

What about Short Games?



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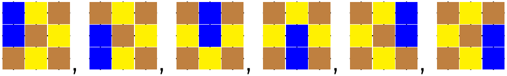

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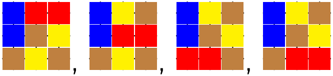

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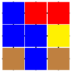
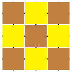


No Right move! (Left wins)

What about Short Games?



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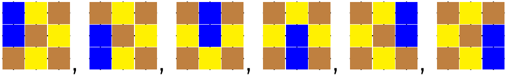

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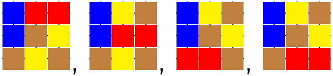

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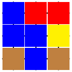
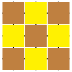


No Right move! (Left wins) Go back and change Right's last move...

What about Short Games?



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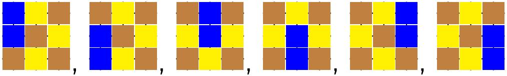

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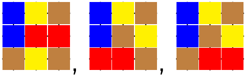

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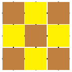

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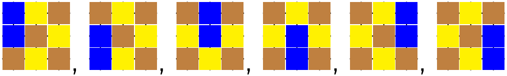

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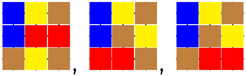

- Continue to decide $G \in \mathcal{L} \cup \mathcal{N}$

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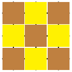

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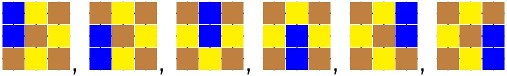

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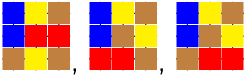

- Continue to decide $G \in \mathcal{L} \cup \mathcal{N}$
- How many rows of boards do I need?

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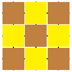

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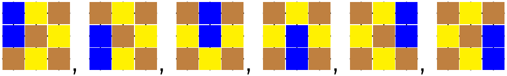

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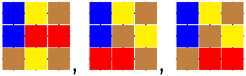

- Continue to decide $G \in \mathcal{L} \cup \mathcal{N}$
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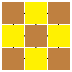

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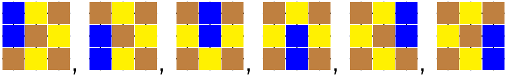

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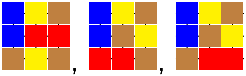

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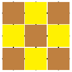

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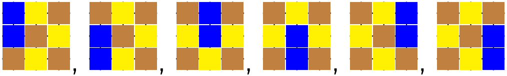

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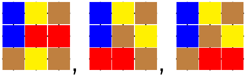

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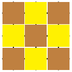

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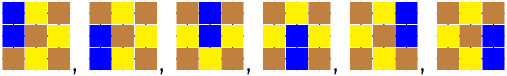

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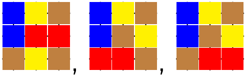

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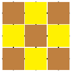
What about Short Games?



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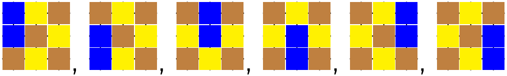

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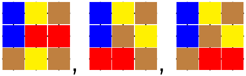

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Example: Impartial Col is PSPACE-hard

Impartial Col: 2-coloring placement game

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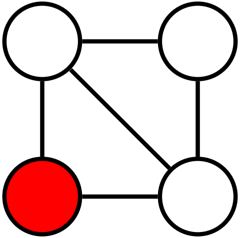

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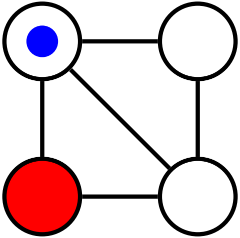

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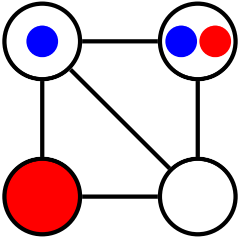

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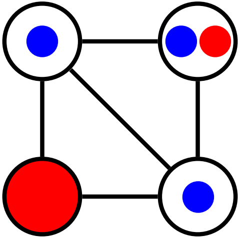

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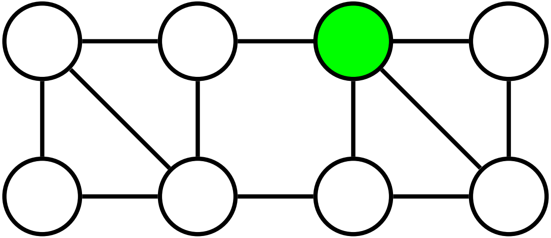

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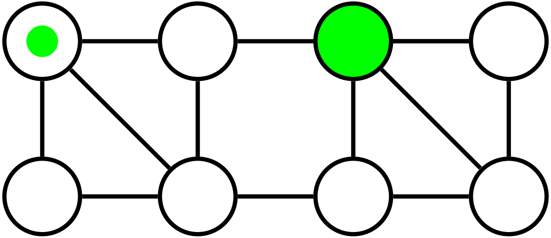

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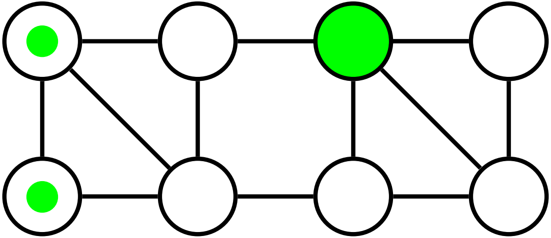

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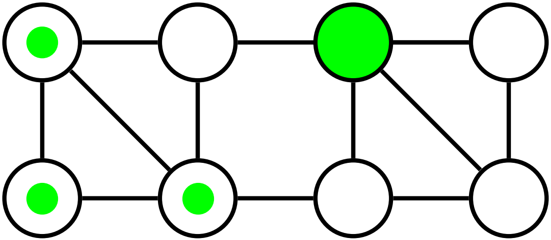

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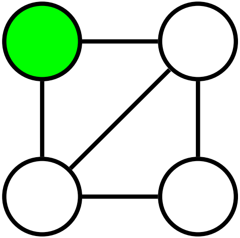

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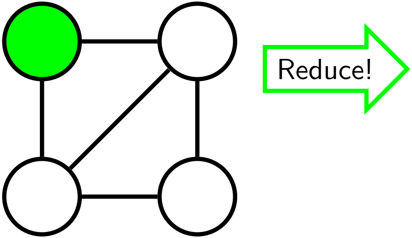

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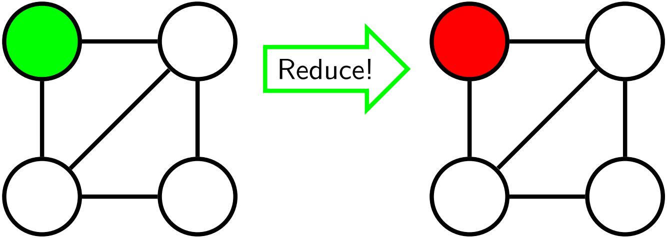

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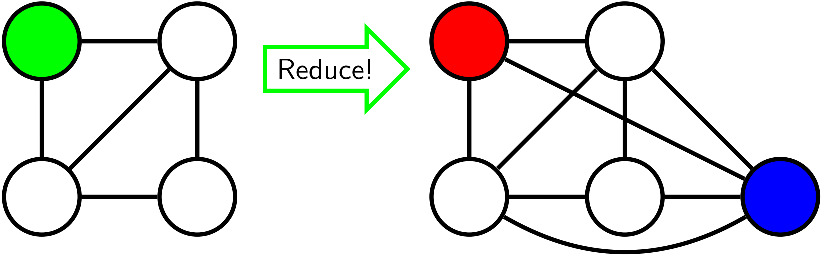

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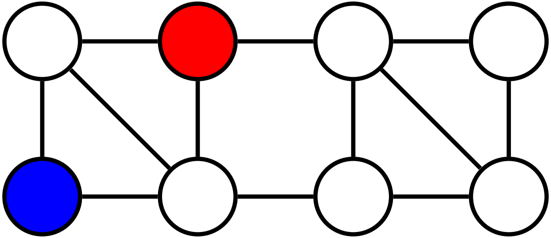

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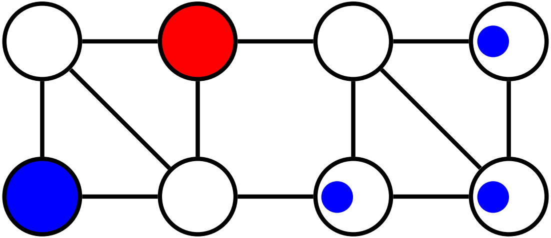

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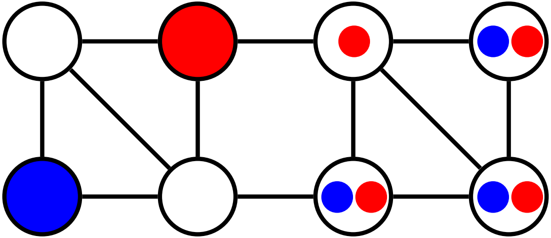

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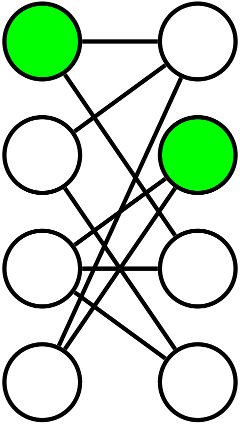

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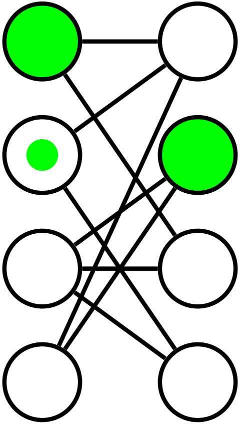

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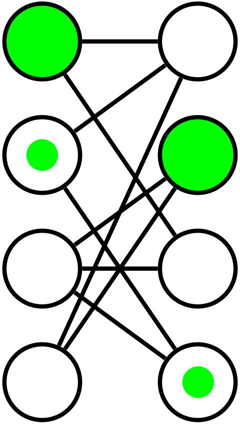

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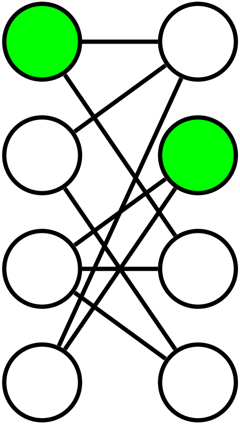

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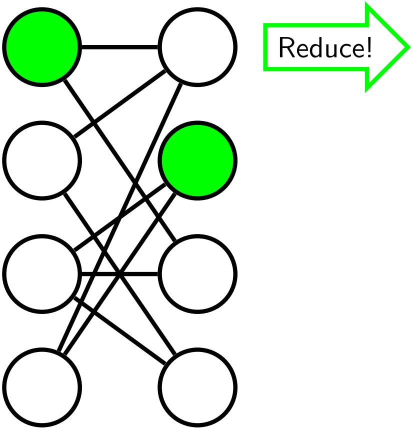

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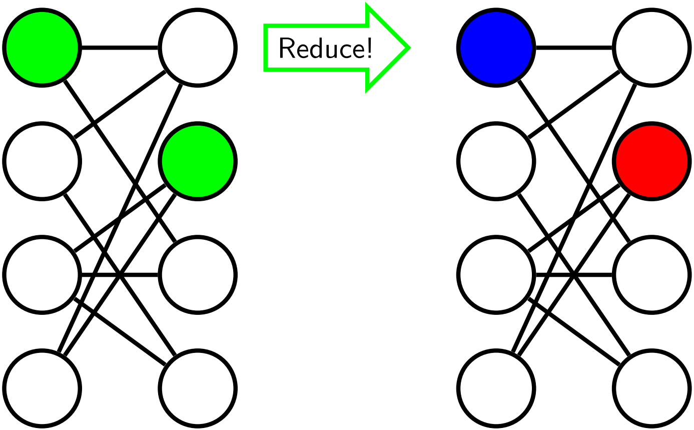

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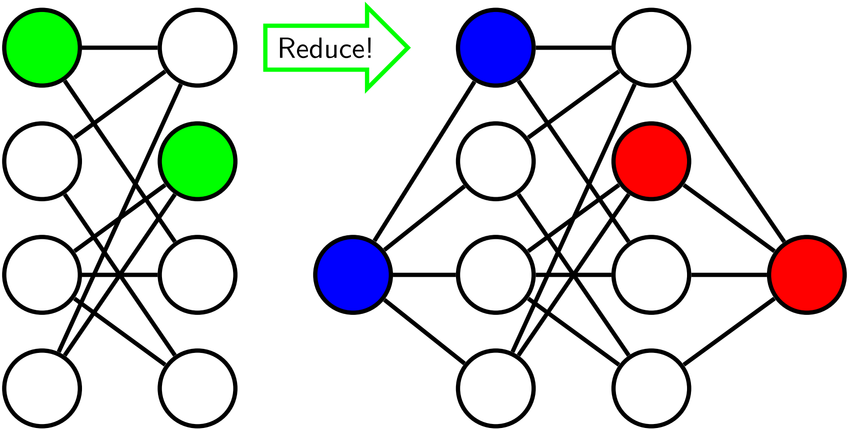

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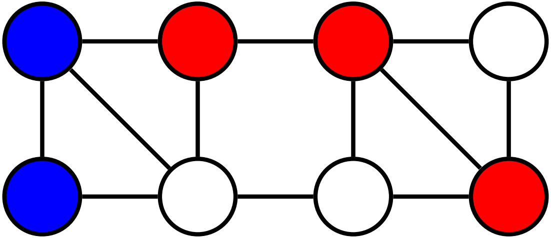

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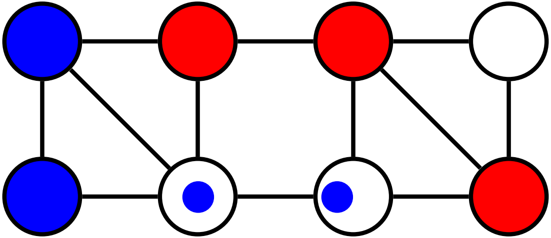

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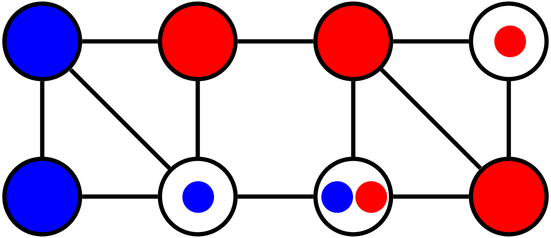

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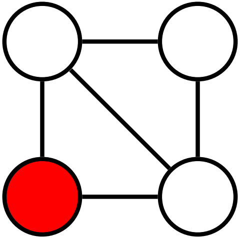

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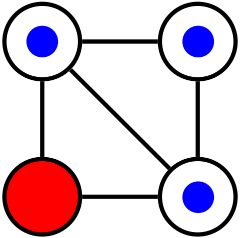

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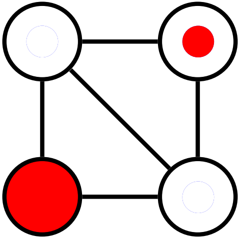

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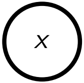

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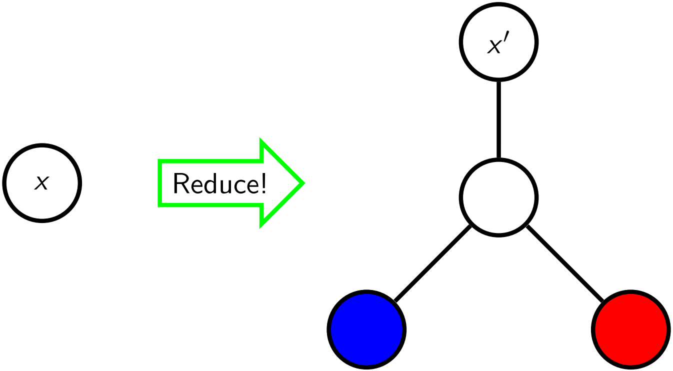

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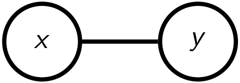

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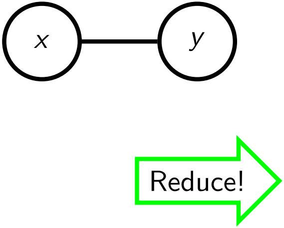

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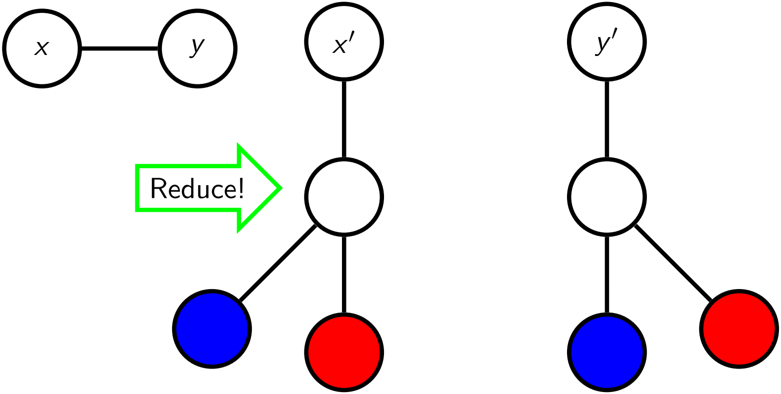

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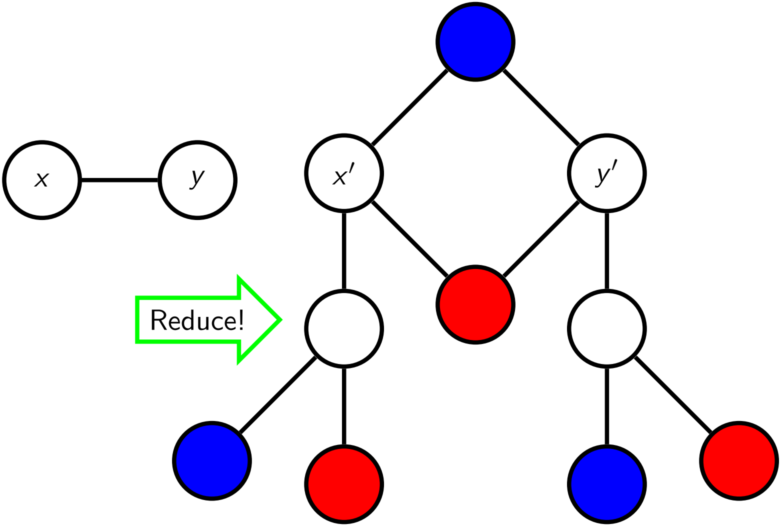

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Some of the first PSPACE-hard games were these, proven from QSAT.

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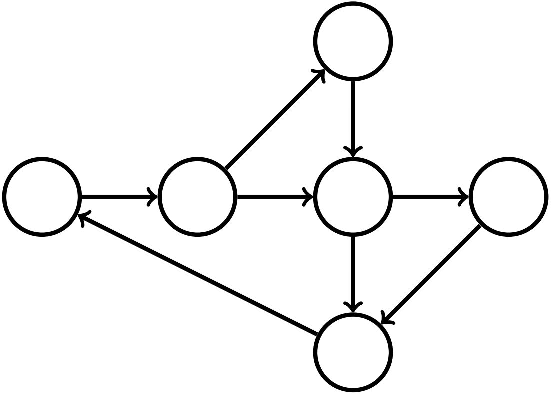

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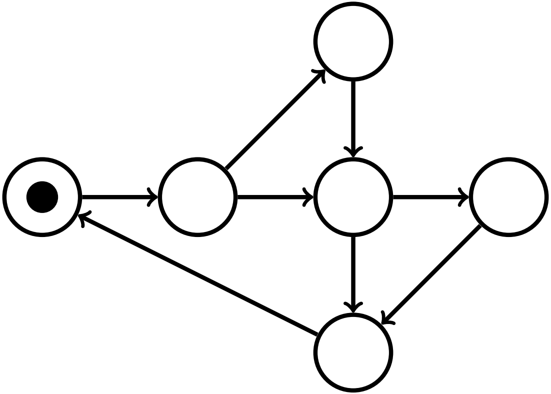

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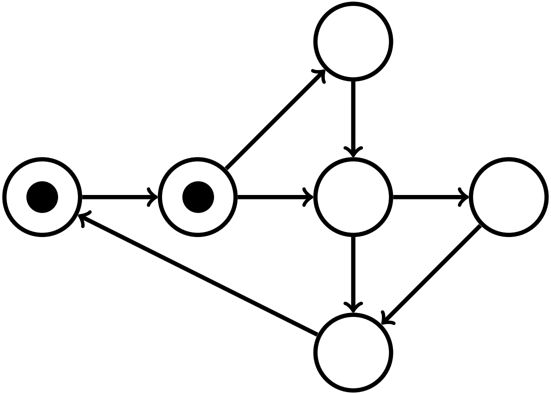

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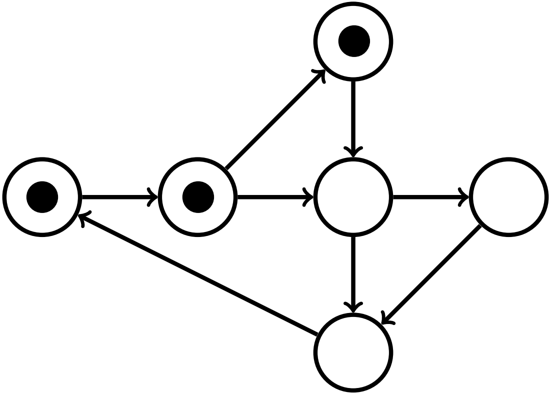

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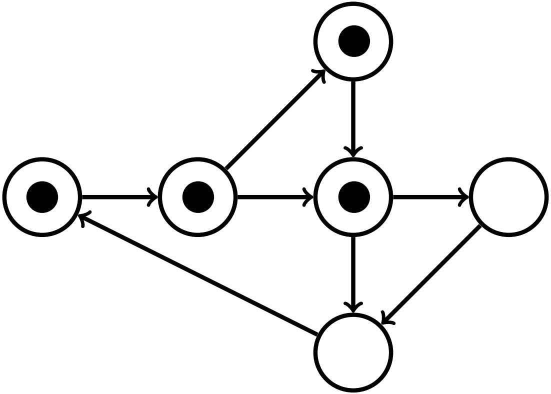

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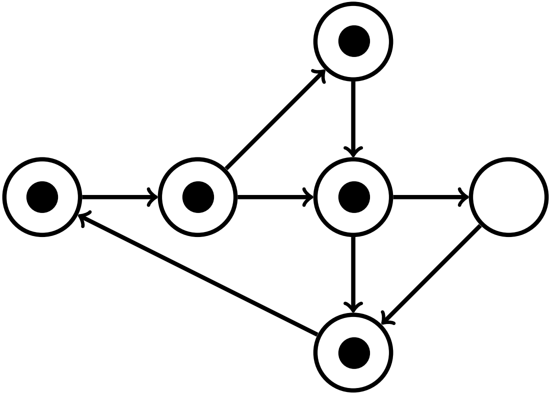

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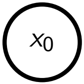

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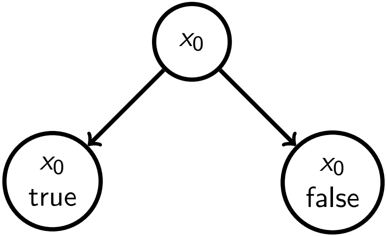

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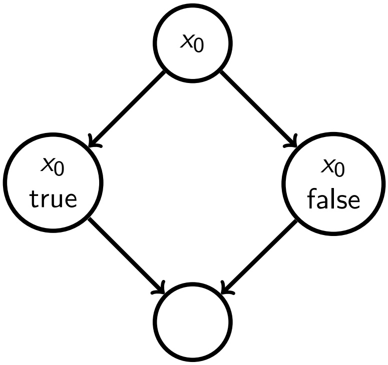

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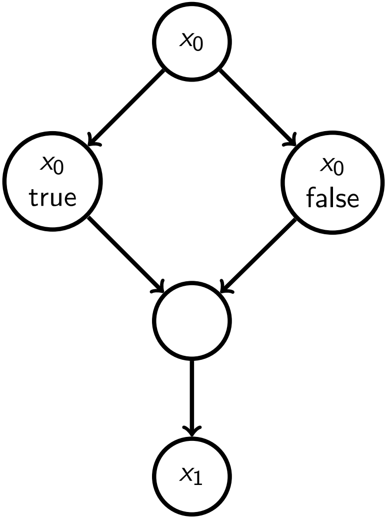

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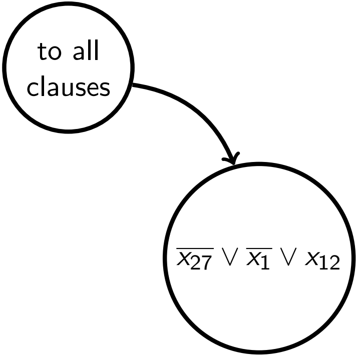

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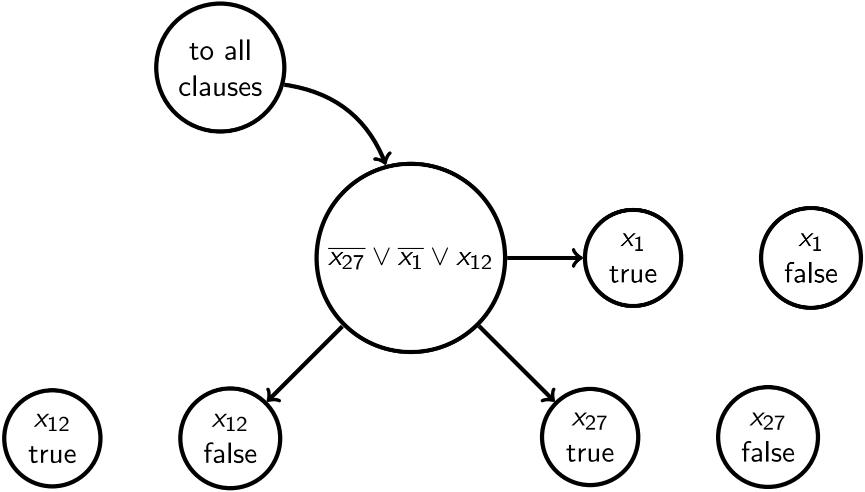

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- Usually, starting positions are easy.

Specific Board Geometry

"Snapping to a grid" is difficult.

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[6] Few slides back.
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- This seems so backwards! Usually the general case is strongest!

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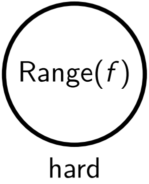

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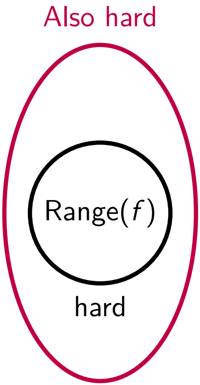

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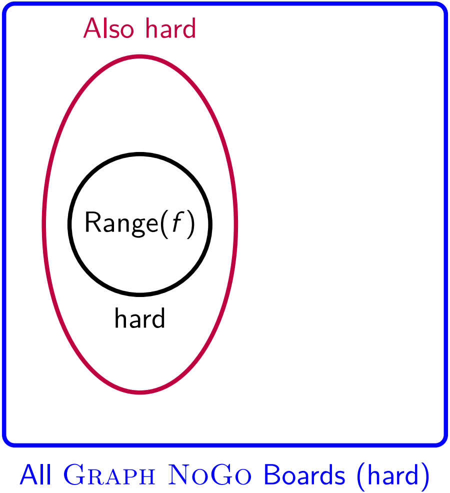

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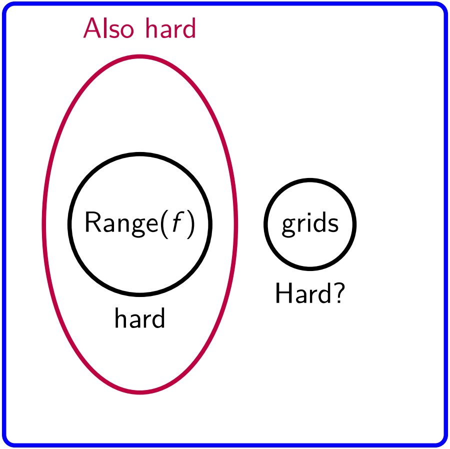

All Graph NoGo Boards (hard)

State of the Art

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Thank you!

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Extra thanks to Dan Burgess and Matt Ferland for proof-watching early versions of this talk.