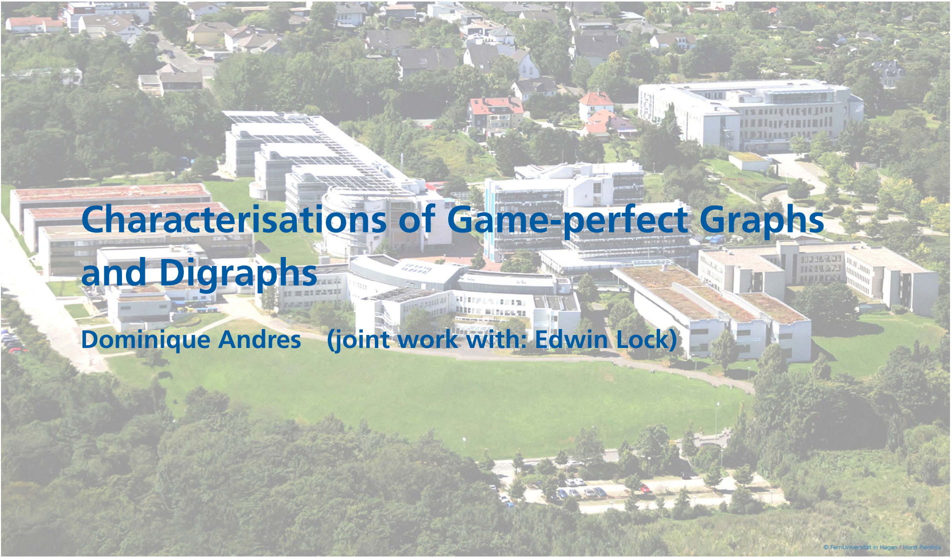

Characterisations of Game-perfect Graphs and Digraphs

Dominique Andres (joint work with: Edwin Lock)

Contents

1.

Game-perfect undirected graphs

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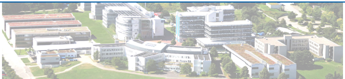

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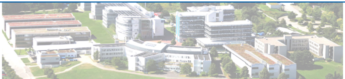

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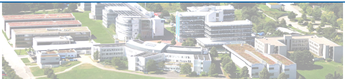

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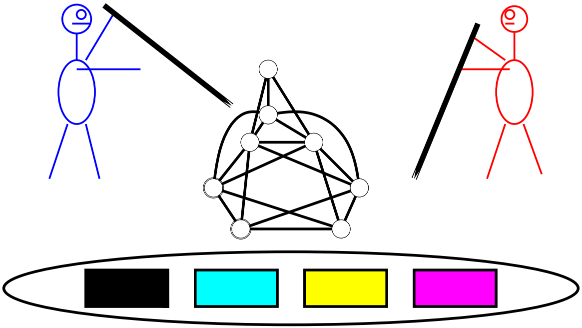

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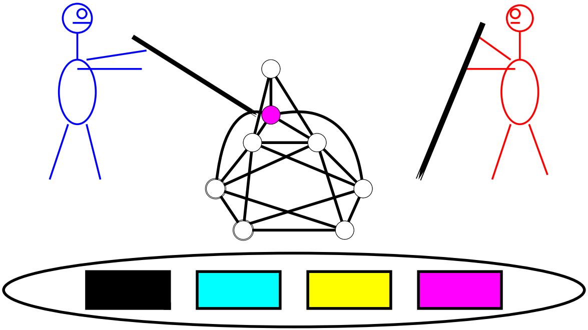

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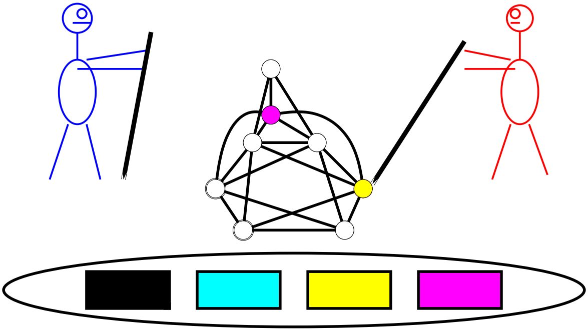

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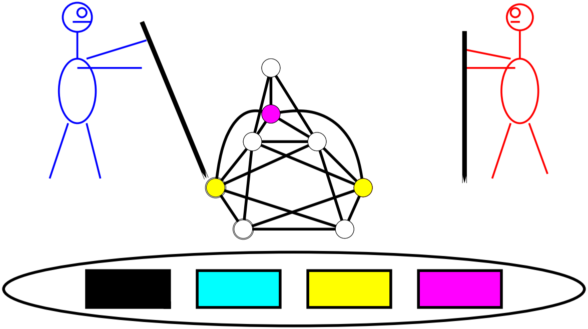

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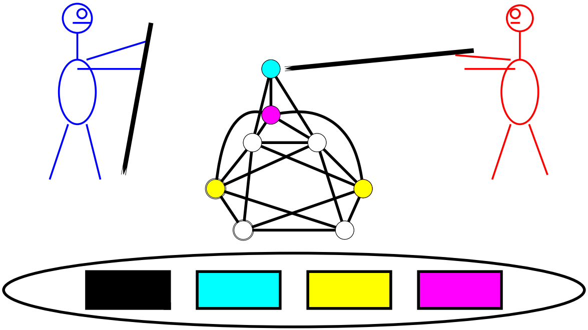

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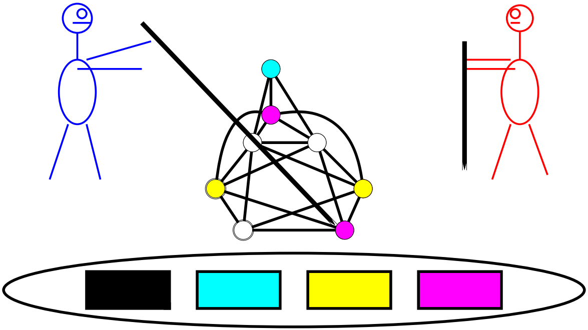

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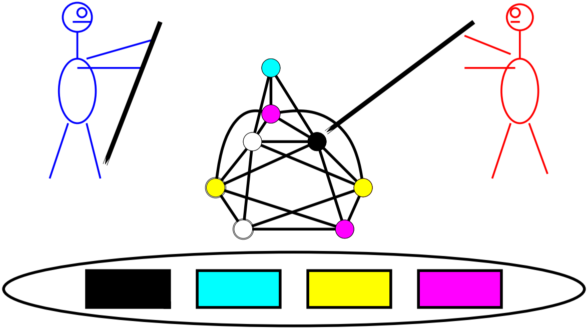

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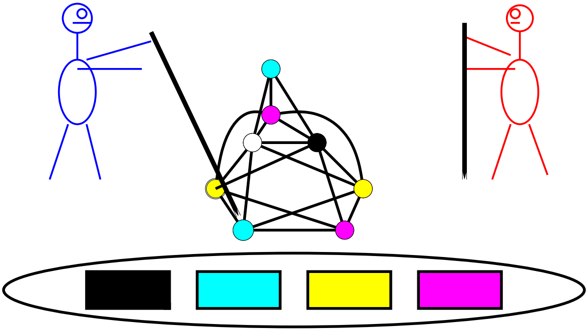

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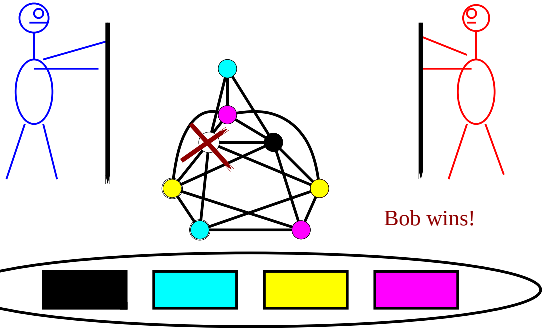
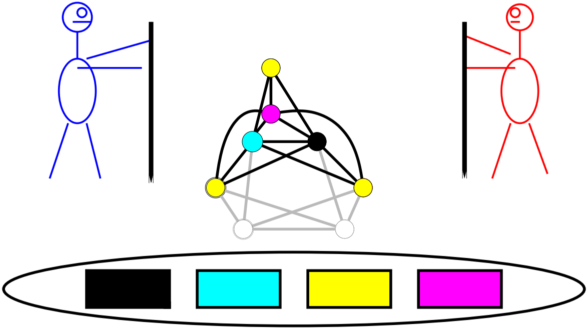
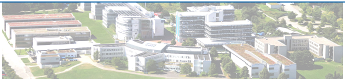


Alice and Bob play the game ... Alice has no chance!



6 types of games

	Alice begins	Bob begins
Alice may pass	game $[A, A]$	game $[B, A]$
missing a turn not allowed	game $[A, -]$	game $[B, -]$
Bob may pass	game $[A, B]$	game $[B, B]$



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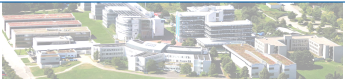

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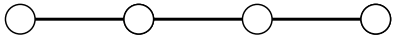

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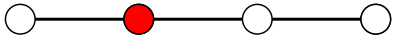

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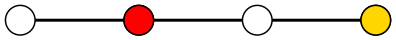

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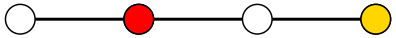

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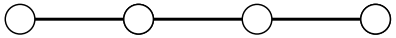

game $[B, -]$

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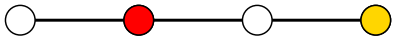

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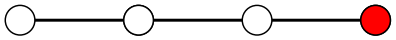

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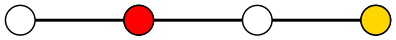

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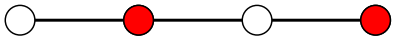

game $[B, -]$

Alice wins! $\chi_{[B,-]}(P_4) = 2$

Game-perfect graphs

Let $g = [X, Y]$ be one of the 6 games.

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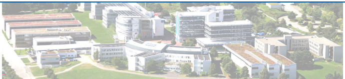

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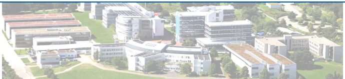

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Remark. Game-perfect graphs are special cases of perfect graphs.

6 classes of game-perfect graphs

$[B, B]$-perfect graphs: structural characterisation A. 2012
$[A, B]$-perfect graphs: structural characterisation A. 2012
$[A, -]$-perfect graphs: structural characterisation A. 2012
$[B, -]$-perfect graphs: structural characterisation with Edwin Lock 2016+
$[B, A]$-perfect graphs: open problem (no characterisation known)
$[A, A]$-perfect graphs: open problem (no characterisation known)

What does such a characterisation look like?

Theorem

Let G be a graph.

(i) G is game-perfect if and only if

(ii) G does not contain any of the forbidden structures F_j (left).

This is the case if and only if

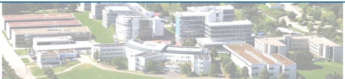
(iii) G belongs to one of the structural types E_i (right).

forbidden induced subgraphs:

$F_1, \ldots, F_n$

allowed structures:

$E_1, \ldots, E_m$



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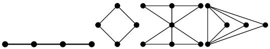
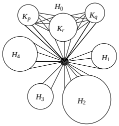
4 forbidden induced subgraphs:



1 allowed structure:



$\times k$

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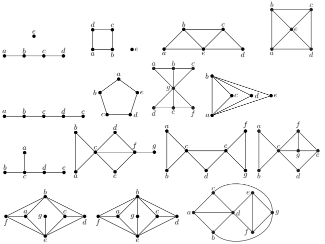
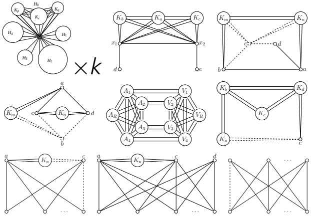
15 forbidden induced subgraphs:



9 allowed structures:



Sketch of proof technique:

(i)$\Longrightarrow$(ii): Prove: Bob wins on any forbidden configuration F_j.

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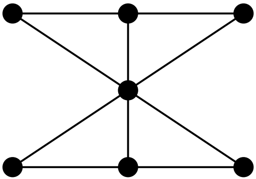
(ii)$\Longrightarrow$(iii): Structural characterisation of graphs not containing any F_j (this is the hard part of the proofs!)

(iii)$\Longrightarrow$(i): Prove:

1. Alice wins on any allowed structure E_i
2. Every substructure of an allowed structure E_i is again an allowed structure E_{i_0}

Example for (i)$\Longrightarrow$(ii): Bob wins on “double fan”



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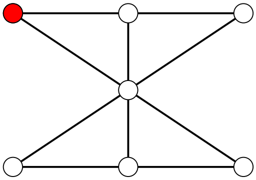

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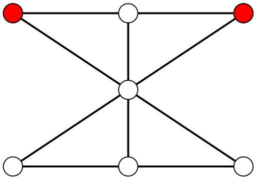

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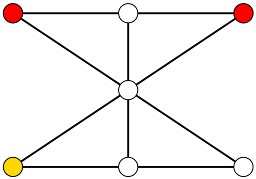

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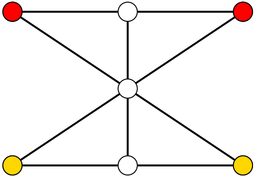

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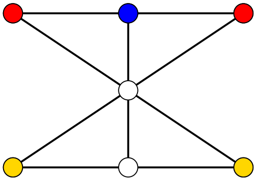
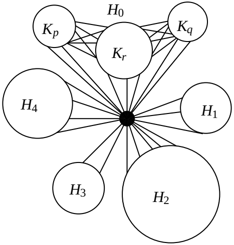
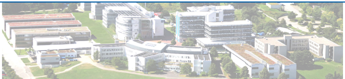


Example for (iii)$\Longrightarrow$(i): Alice wins on "ear animal"





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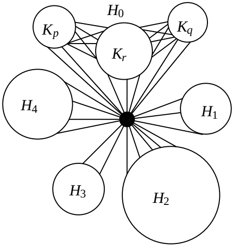

Main goal is that the "ears" contain the same colours
unless the second ear is fully coloured.
And Alice should colour the central vertex as fast as possible.

Idea of proof of (ii)$\Longrightarrow$(iii): Structural characterisation

Structure of (nontrivial)
$[B, B]$-perfect graphs
(A. 2012):

Structure of (connected)
$[B, -]$-perfect graphs
(with Edwin Lock 2016+):

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Structure of (nontrivial)
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$\longrightarrow$ dominating vertex exists
(by Lemma of Wolk (1965))

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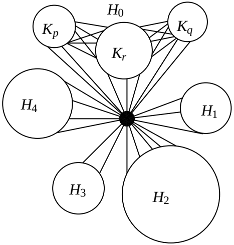

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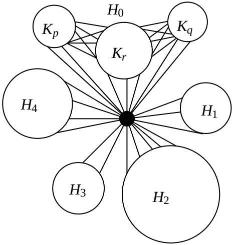

inner structure simple
(1 page of case distinctions)

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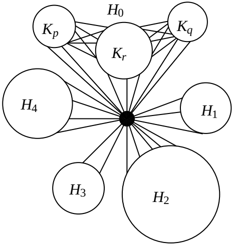

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(Cozzens&Kelleher (1990))

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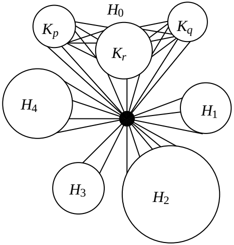

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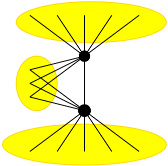
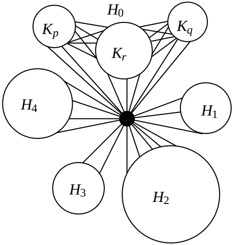
$\longrightarrow$ dominating edge exists
(Cozzens&Kelleher (1990))



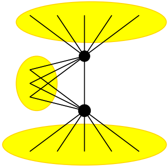
Idea of proof of (ii)$\Longrightarrow$(iii): Structural characterisation

Structure of (nontrivial) $[B, B]$-perfect graphs (A. 2012):
$\longrightarrow$ dominating vertex exists (by Lemma of Wolk (1965))



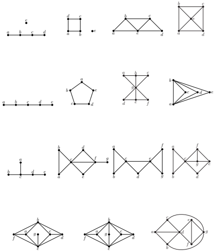
inner structure simple
(1 page of case distinctions)

Structure of (connected) $[B, -]$-perfect graphs (with Edwin Lock 2016+):
$\longrightarrow$ dominating edge exists (Cozzens&Kelleher (1990))



—inner structure simple
—examine structure of adjacencies between the three parts
(20 pages of case distinctions)

Forbidden induced subgraphs for game-perfectness



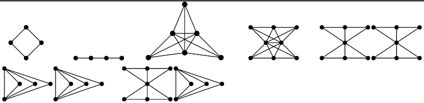
Lock (2016) $[B, -]$

$[B, A]$

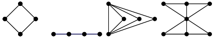
?

$[A, A]$

?



A. (2012) $[A, -]$, $[A, B]$

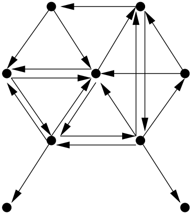


A. (2012) $[B, B]$

2.

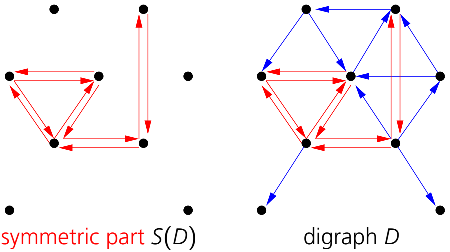
Digraphs

Digraphs

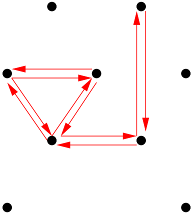


digraph D

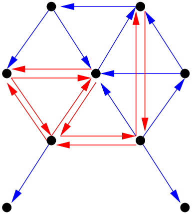
Digraphs



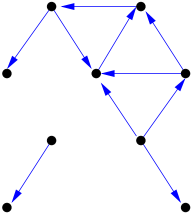
Digraphs



symmetric part $S(D)$



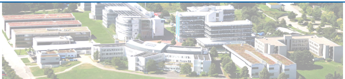
digraph D



oriented part $O(D)$

Dichromatic number, symmetric cliques, perfect digraphs

The dichromatic number $\chi(D)$ of a digraph D is the smallest number of induced acyclic subdigraphs of D that cover the vertices of D. [Neumann-Lara 1982]



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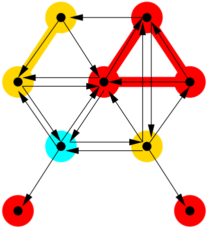
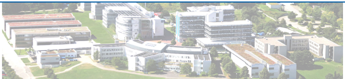
$\longrightarrow$ **no monochromatic directed cycles!**





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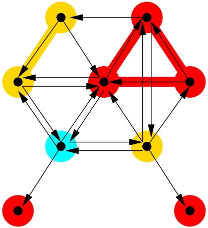
A symmetric clique is a complete digraph (without loops) identical to its symmetric part.

$\omega(D)$ = size of largest symmetric clique.



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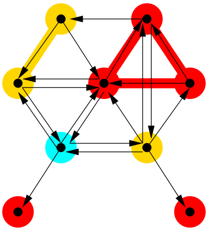


A digraph D is perfect if,

for any induced subdigraph H of D,

$$\omega(H) = \chi(H).$$

A generalization of the Strong Perfect Graph Theorem

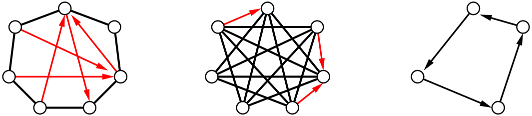
Theorem (A.&Hochstättler(2015))

A digraph is perfect if and only if it does not contain induced subdigraphs of the following types:

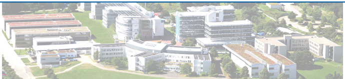
(1) *filled odd holes: i.e. D with $S(D)$ is odd hole resp.*

(2) *filled odd antiholes: i.e. D with $S(D)$ is odd antihole resp.*

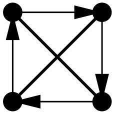
(3) *directed cycles of length ≥ 3.*

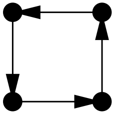


type (1) type (2) type (3)



Perfectness is non-closed by taking complements

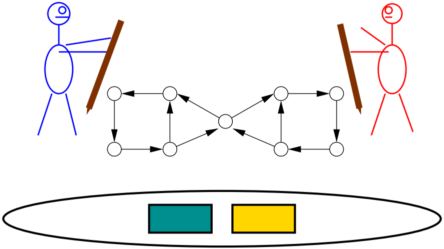

perfect


non perfect

2. a)

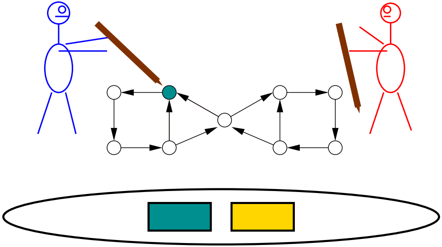
Weakly game-perfect digraphs

Weak digraph colouring games [Yang&Zhu(2010)]



Alice and Bob alternately colour uncoloured vertices of D with a colour from the set C, such that they do not create monochromatic directed cycles.
Alice wins if every vertex is coloured.

Weak digraph colouring games [Yang&Zhu(2010)]



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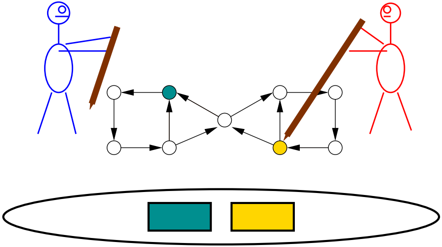

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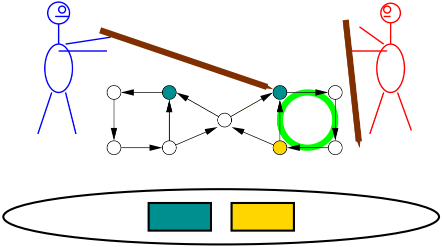

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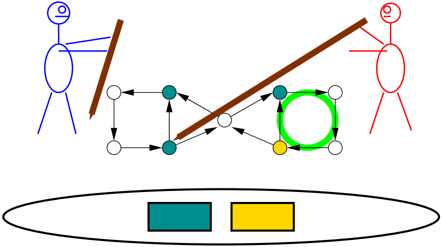

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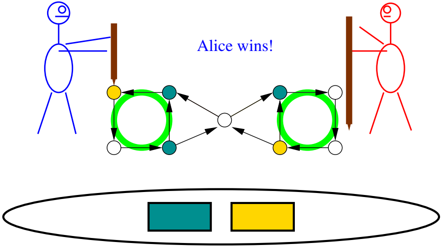


Alice and Bob alternately colour uncoloured vertices of D with a colour from the set C, such that they do not create monochromatic directed cycles.
Alice wins if every vertex is coloured.

Weakly game-perfect digraphs

Definition (weak game chromatic number)

The weak game chromatic number $\chi_{wg}(D)$ of a digraph D is the smallest number of colours so that Alice has a winning strategy for the weak colouring game played on D.

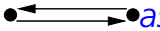
Definition (weakly game-perfect digraph)

A digraph D is weakly game-perfect (or *wg*-perfect) if, for any induced subdigraph H of D,

$$\omega(H) = \chi_{wg}(H).$$

Characterisation of weakly game-perfect digraphs

Lemma

If D does not contain an induced cycle $\vec{C}_n$, $n \geq 3$, then every directed cycle has a (symmetric) edge  *as a chord.*

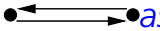
Theorem

For a weak game wg and the associated undirected game g, a digraph D is wg-perfect if and only if

1. *its symmetric part $S(D)$ is a g-perfect graph and*
2. *D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as an induced subdigraph.*

Characterisation of weakly game-perfect digraphs

Lemma

If D does not contain an induced cycle $\vec{C}_n$, $n \geq 3$, then every directed cycle has a (symmetric) edge  *as a chord.*

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1. *its symmetric part $S(D)$ is a g-perfect graph and*
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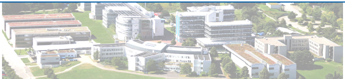
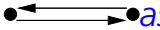
If 2. is not true: *D* is not perfect, thus *D* is not *wg*-perfect.



Characterisation of weakly game-perfect digraphs

Lemma

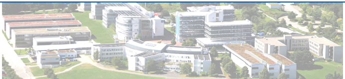
If D does not contain an induced cycle $\vec{C}_n$, $n \geq 3$, then every directed cycle has a (symmetric) edge  *as a chord.*

Theorem

For a weak game wg and the associated undirected game g, a digraph D is wg-perfect if and only if

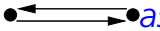
1. *its symmetric part $S(D)$ is a g-perfect graph and*
2. *D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as an induced subdigraph.*

If 2. is true but 1. is not true, i.e. $S(D)$ is non-g-perfect.
$\Rightarrow$ $\exists$ subgraph H, so that Bob wins on $S(H)$.
He uses the same strategy for the play on H.
Whenever he would close a monochromatic directed cycle, by Lemma this cycle would be monochromatic edge or have a monochromatic edge as a chord, contradicting the fact that he has a strategy for $S(H)$. $\Rightarrow$ D is not wg-perfect.



Characterisation of weakly game-perfect digraphs

Lemma

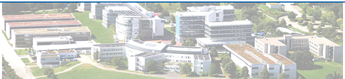
If D does not contain an induced cycle $\vec{C}_n$, $n \geq 3$, then every directed cycle has a (symmetric) edge  *as a chord.*

Theorem

For a weak game wg and the associated undirected game g, a digraph D is wg-perfect if and only if

1. *its symmetric part* $S(D)$ *is a g-perfect graph and*
2. *D does not contain any directed cycle* $\vec{C}_n$ *with* $n \geq 3$ *as an induced subdigraph.*

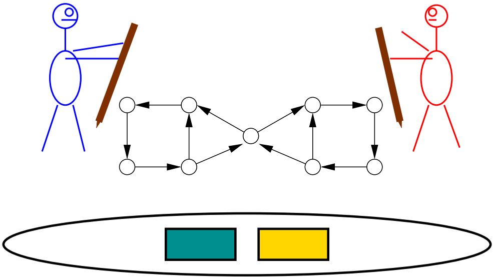
If 1. and 2. are true: Let H be induced subdigraph of D.
$\Rightarrow$ Alice has winning strategy on $S(H)$.
She uses the same strategy for the play on H.
Whenever she would close a monochromatic directed cycle, by Lemma this cycle would be monochromatic edge or have a monochromatic edge as a chord, contradicting the fact that she has a strategy for $S(H)$. $\Rightarrow$ D is wg-perfect.



2. b)

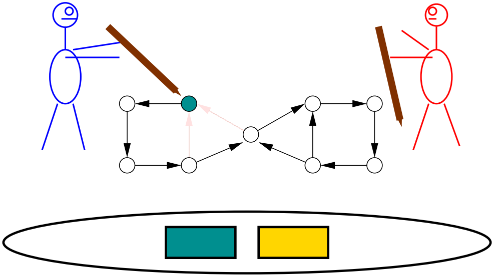
Strongly game-perfect digraphs

Strong digraph colouring games



Alice and Bob alternately colour uncoloured vertices of *D* with a colour from the set *C*, which is different from colours of its **in**-neighbours.
Alice wins if every vertex is coloured.
Bob wins if a vertex is surrounded by all colours.

Strong digraph colouring games



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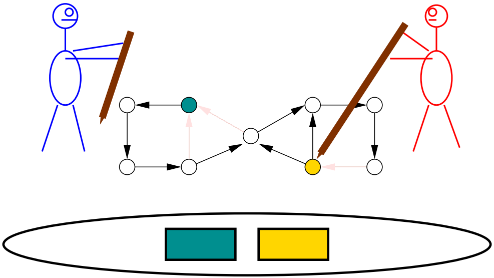

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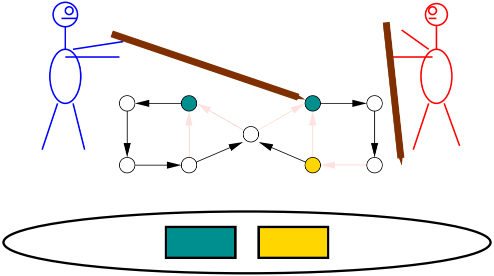

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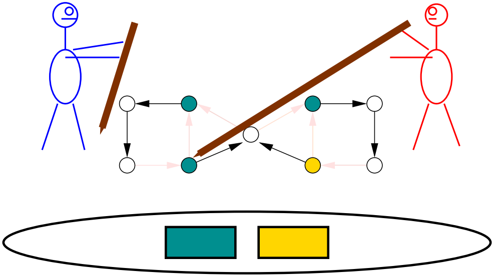

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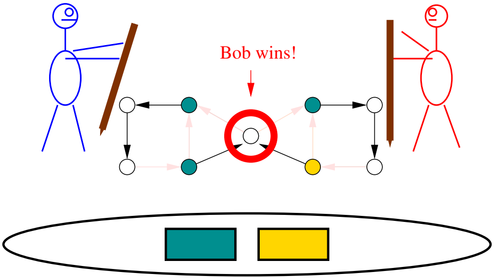
Strong digraph colouring games



Alice and Bob alternately colour uncoloured vertices of D with a colour from the set C, which is different from colours of its **in**-neighbours.
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Strongly game-perfect digraphs

Definition (game chromatic number)

The (strong) game chromatic number $\chi_g(D)$ of a digraph D is the smallest number of colours so that Alice has a winning strategy for the strong colouring game played on D.

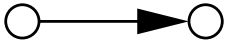
Definition (strongly game-perfect digraph)

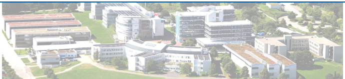
A digraph D is (strongly) game-perfect (or g-perfect) if, for any induced subdigraph H of D,

$$\omega(H) = \chi_g(H).$$

[6 types of games]

Trivial if Bob begins: No additional structures





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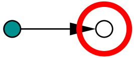

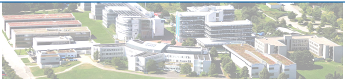

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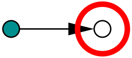
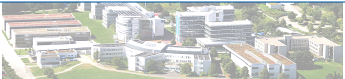


A single arc is not $[B, *]$-perfect.



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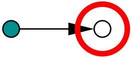


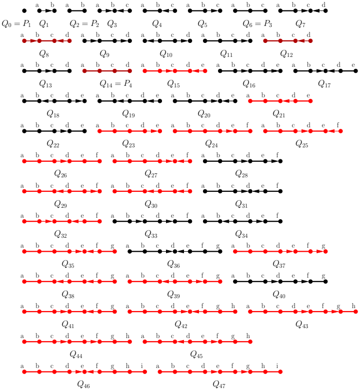
A single arc is not $[B, *]$-perfect.

Theorem

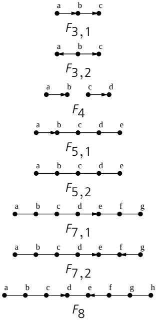
For the games, where Bob begins, the class of game-perfect digraphs is equal to its subclass the class of game-perfect undirected graphs.

Nontrivial if Alice begins: game $[A, A]$ on paths

The 48 $[A, A]$-perfect paths
(red: non $[A, -]$/non $[A, B]$-perfect)

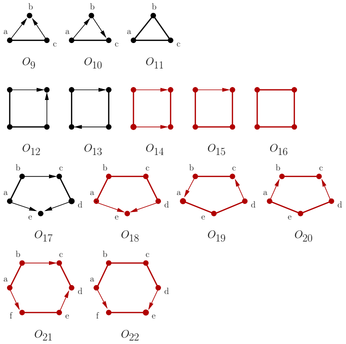


All minimal forbidden paths
for the game $[A, A]$

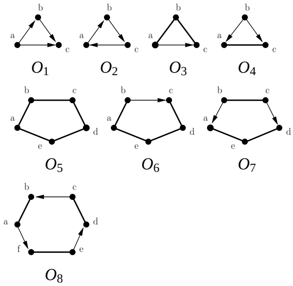


...if Alice begins: game $[A, A]$ on cycles

The 14 $[A, A]$-perfect cycles
(red: non $[A, -]$/non $[A, B]$-perfect)



Forbidden cycles
for the game $[A, A]$

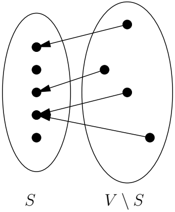


Observation

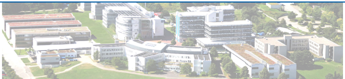
Let C be a cycle with $n \geq 7$ vertices.
Then C is not game-perfect.

Kernels in digraphs

A kernel S of
digraph $D = (V, A)$:



S independent and
absorbing



Kernels in digraphs

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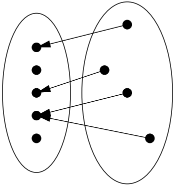

S independent and absorbing

Theorem (A.&Hochstättler(2015))

The complement of a perfect digraph has a kernel.

Kernels in digraphs

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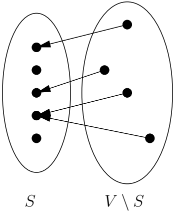


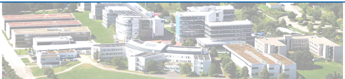
S independent and absorbing

Theorem (A.&Hochstättler(2015))

The complement of a perfect digraph has a kernel.

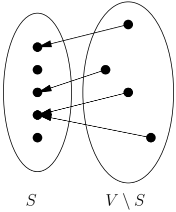
Theorem

$[A, -]$*-perfect digraphs are the complement of a perfect digraph.*



Kernels in digraphs

A kernel S of digraph $D = (V, A)$:



S independent and absorbing

Theorem (A.&Hochstättler(2015))

The complement of a perfect digraph has a kernel.

Theorem

$[A, -]$-perfect digraphs are the complement of a perfect digraph.

Corollary

$[A, -]$-perfect digraphs have a kernel.

Kernels in $[A, A]$-perfect digraphs

We can show similarly

Theorem

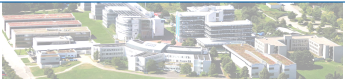
Every $[A, A]$-perfect digraph D that does not contain the complement of a directed cycle $\vec{C}_4$ has a kernel.

(a) (b)

$[A, A]$-perfect digraphs: (a) does not have a kernel (b) has a kernel

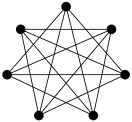
Open Question

Characterise the $[A, A]$-perfect digraphs that contain a $\vec{C}_4^C$: which of them have a kernel?



The remaining undirected cases

$[B, A]$- resp. $[A, A]$-perfect graphs cannot be described by a finite list of minimal forbidden configurations, since every odd antihole is a minimal forbidden configuration.



Open Question

Characterize $[B, A]$- and $[A, A]$-perfect undirected graphs.

On characterizing game-perfect digraphs

For the games, where Alice begins, we have the following open problems.

Problem

Characterize (strongly) game-perfect superorientations of trees.

On characterizing game-perfect digraphs

For the games, where Alice begins, we have the following open problems.

Problem

Characterize (strongly) game-perfect superorientations of trees.

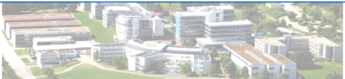
Problem

Characterize (strongly) game-perfect digraphs by a complete list of forbidden configurations.

Complexity

Open Question

What is the complexity of the (di)graph colouring games?



Thank you

Thank you for your attention.

Open Questions on Game-perfect Graphs and Digraphs

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FernUniversität in Hagen

Games and Graphs Workshop, 23–25 October 2017

Question 1: Finitely many minimal forbidden trees?

A digraph D is $[A, A]$-perfect if, for any induced subdigraph, the symmetric clique number equals the game chromatic number for the game where Alice may have the first move and missing a turn is allowed for Alice.

Conjecture

The number of minimal forbidden semiorientations of trees concerning $[A, A]$-perfect digraphs is finite.

- there are infinitely many forbidden configurations: e.g. all odd antiholes
- digraphs with diameter ≥ 9 are not $[A, A]$-perfect
- there are infinitely many $[A, A]$-perfect semiorientations of trees: e.g. all in-stars

Question 2: The 5-classes-conjecture

A digraph D is $[A,-]$-perfect ($[A,B]$-perfect) if, for any induced subdigraph, the symmetric clique number equals the game chromatic number for the game where Alice may have the first move and missing a turn is not allowed (allowed for Bob).

Conjecture

For any digraph D,

$$D \text{ is } [A,-]\text{-perfect} \Longleftrightarrow D \text{ is } [A,B]\text{-perfect.}$$

- "$\Longleftarrow$" is true, trivially
- true for undirected graphs
- true for semiorientations of cycles resp. forests of paths

Question 3: Characterize χ-col-perfect graphs

A graph G is χ-col-perfect if, for any induced subgraph, the game chromatic number equals the game colouring number.

A graph G is col-perfect if, for any induced subgraph, the clique number equals the game colouring number.

A graph G is game-perfect if, for any induced subgraph, the game chromatic number equals the clique number.

Problem

Characterize the class of χ-col-perfect graphs.

- col-perfectness has been characterized for every game
- game-perfectness has been characterized partially (for 4 out of 6 games)
- col-perfect = game-perfect $\cap$ χ-col-perfect