Ryūō Nim: A Variant of the classical game of Wythoff's Nim

Tomoaki Abuku, Masanori Fukui, Ryohei Miyadera, Yushi Nakaya, Kouki Suetsugu, Yuki Tokuni

Graduate School of Pure and Applied Sciences, University of Tsukuba, Japan

Games and Graphs Workshop
(Lyon, 23–25 October, 2017)

Contents

Wythoff's Nim

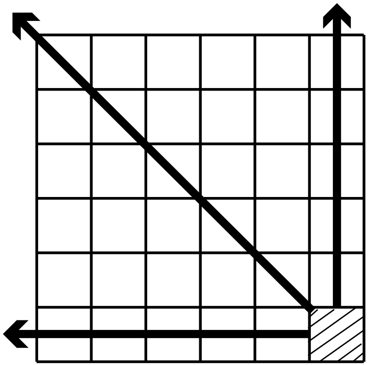
Wythoff's Nim is a well-known impartial game with two heaps of tokens. The rules are as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
- remove the same number of tokens from both heaps.

The end position is the state of no tokens in both heaps. Wythoff's Nim is also called "Corner the Queen."

Corner the Queen

The rules of the corner the queen are as follows:
Each player, when it is his turn to move, can move a Chess queen an arbitrary distance North, West or North-West as indicated by arrows.



Clearly, this game is equivalent to Wythoff's Nim.

Wythoff's Nim

The Grundy value of Wythoff's Nim position is not known, but the following theorem is well-known about $\mathcal{P}$-positions of Wythoff's Nim.

Theorem

Let (m, n) $(m \leq n)$ be a Wythoff's Nim position.
For $n - m = k$, the $\mathcal{P}$-positions of Wythoff's Nim are given by

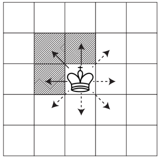
$$(\lfloor k\Phi \rfloor, \lfloor k\Phi \rfloor + k), (\lfloor k\Phi \rfloor + k, \lfloor k\Phi \rfloor),$$

where Φ is the golden ratio, i.e. $\Phi = \frac{1+\sqrt{5}}{2}$.

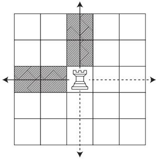
Ryūō Nim

Movement of pieces in Chess

- King; can move one by one, vertically, horizontally and diagonally.



- Rook; can move as many steps as you like, vertically and horizontally.

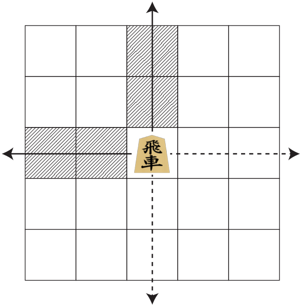


There are other pieces of chess, but this time I will only consider these two.

Ryūō Nim

Movement of pieces in Shōgi (Japanese chess)
Shōgi is a Japanese board game similar to Chess.
In Shōgi, the movement of the pieces are almost the same with that of Chess.

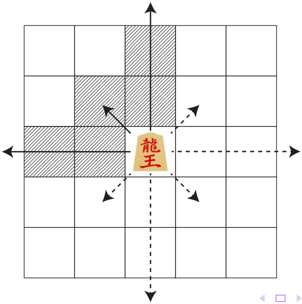
- Hisya ("flying chariot"); the movement is exactly the same with that of Rook.



Ryūō Nim

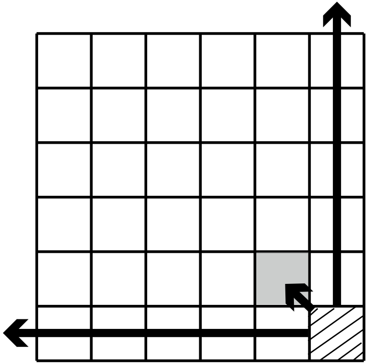
In Chess, when a piece called pawn reaches the first row, it is replaced by a piece of the player's choice (promotion).
In Shōgi, some of the pieces turn over and become more powerful when they reach the third row.
For example, in the case of a Hisya, it turns over and becomes a Ryūō, which is more powerful than a Hisya.

- Ryūō ("dragon king", promoted Hisya); can move both the Hisya and the king.



Ryūō Nim

Ryūō Nim is equivalent to the game played with a Ryūō instead of a queen in "Corner the Queen." The legal move is to remove any number of tokens from a single heap (as in Nim) or remove one token from both heaps.



The Grundy value of Ryūō Nim

Grundy values of Ryūō Nim are examined and they are shown in the following table.

- The table of the Grundy value of Ryūō Nim

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14	12	16	17	15	19	20	18
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12	13	17	15	16	20	18	19
3	3	4	5	0	1	2	9	10	11	6	7	8	15	16	17	12	13	14	21	22	23
4	4	5	3	1	2	0	10	11	9	7	8	6	16	17	15	13	14	12	22	23	21
5	5	3	4	2	0	1	11	9	10	8	6	7	17	15	16	14	12	13	23	21	22
6	6	7	8	9	10	11	0	1	2	3	4	5	18	19	20	21	22	23	12	13	14
7	7	8	6	10	11	9	1	2	0	4	5	3	19	20	18	22	23	21	13	14	12
8	8	6	7	11	9	10	2	0	1	5	3	4	20	18	19	23	21	22	14	12	13
9	9	10	11	6	7	8	3	4	5	0	1	2	21	22	23	18	19	20	15	16	17
10	10	11	9	7	8	6	4	5	3	1	2	0	22	23	21	19	20	18	16	17	15
11	11	9	10	8	6	7	5	3	4	2	0	1	23	21	22	20	18	19	17	15	16
12	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6	7	8
13	13	14	12	16	17	15	19	20	18	22	23	21	1	2	0	4	5	3	7	8	6
14	14	12	13	17	15	16	20	18	19	23	21	22	2	0	1	5	3	4	8	6	7
15	15	16	17	12	13	14	21	22	23	18	19	20	3	4	5	0	1	2	9	10	11
16	16	17	15	13	14	12	22	23	21	19	20	18	4	5	3	1	2	0	10	11	9
17	17	15	16	14	12	13	23	21	22	20	18	19	5	3	4	2	0	1	11	9	10
18	18	19	20	21	22	23	12	13	14	15	16	17	6	7	8	9	10	11	0	1	2
19	19	20	18	22	23	21	13	14	12	16	17	15	7	8	6	10	11	9	1	2	0
20	20	18	19	23	21	22	14	12	13	17	15	16	8	6	7	11	9	10	2	0	1

When you observe them thoroughly, you can see regularity.

The Grundy value of Ryūō Nim

That means it is divided into 3 × 3 blocks.

- Table of $((x+y) \bmod 3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
3	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
4	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
5	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
6	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
7	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
8	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
9	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
10	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
11	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
12	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
13	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
14	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
15	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
16	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
17	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
18	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
19	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
20	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

$((x+y) \bmod 3)$ is the remainder obtained when $x+y$ is divided by 3.

The Grundy value of Ryūō Nim

When you add this term to the table, we get the table of the Grundy value of Ryūō Nim

- Table of $((x + y) \bmod 3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14	12	16	17	15	19	20	18
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12	13	17	15	16	20	18	19
3	3	4	5	0	1	2	9	10	11	6	7	8	15	16	17	12	13	14	21	22	23
4	4	5	3	1	2	0	10	11	9	7	8	6	16	17	15	13	14	12	22	23	21
5	5	3	4	2	0	1	11	9	10	8	6	7	17	15	16	14	12	13	23	21	22
6	6	7	8	9	10	11	0	1	2	3	4	5	18	19	20	21	22	23	12	13	14
7	7	8	6	10	11	9	1	2	0	4	5	3	19	20	18	22	23	21	13	14	12
8	8	6	7	11	9	10	2	0	1	5	3	4	20	18	19	23	21	22	14	12	13
9	9	10	11	6	7	8	3	4	5	0	1	2	21	22	23	18	19	20	15	16	17
10	10	11	9	7	8	6	4	5	3	1	2	0	22	23	21	19	20	18	16	17	15
11	11	9	10	8	6	7	5	3	4	2	0	1	23	21	22	20	18	19	17	15	16
12	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6	7	8
13	13	14	12	16	17	15	19	20	18	22	23	21	1	2	0	4	5	3	7	8	6
14	14	12	13	17	15	16	20	18	19	23	21	22	2	0	1	5	3	4	8	6	7
15	15	16	17	12	13	14	21	22	23	18	19	20	3	4	5	0	1	2	9	10	11
16	16	17	15	13	14	12	22	23	21	19	20	18	4	5	3	1	2	0	10	11	9
17	17	15	16	14	12	13	23	21	22	20	18	19	5	3	4	2	0	1	11	9	10
18	18	19	20	21	22	23	12	13	14	15	16	17	6	7	8	9	10	11	0	1	2
19	19	20	18	22	23	21	13	14	12	16	17	15	7	8	6	10	11	9	1	2	0
20	20	18	19	23	21	22	14	12	13	17	15	16	8	6	7	11	9	10	2	0	1

The Grundy value of Ryūō Nim

Definition (Grundy value)

Let G be an impartial game position. The Grundy value $\mathcal{G}(G)$ is defined as

$$\mathcal{G}(G) = \mathrm{mex}\{\mathcal{G}(G') \mid G' \in G\}.$$

Therefore, we found that the Grundy value of Ryūō Nim can be expressed as follows:

Theorem

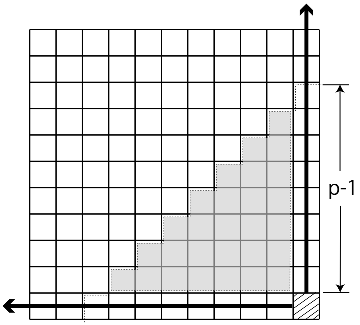
Let (x, y) be a Ryūō Nim position, then we have

$$\mathcal{G}(x, y) = ((x + y) \bmod 3) + 3(\lfloor \tfrac{x}{3} \rfloor \oplus \lfloor \tfrac{y}{3} \rfloor).$$

The Grundy value of Wythoff's Nim position is not known, but we were able to obtain the Grundy value of Ryūō Nim position.

Generalized Ryūō Nim

Restrict the diagonal movement by $p \in \mathbb{Z}_{>1}$.
(The total number of tokens removed from the both heaps at once must be less than p.)



If $p = 3$, then this game is equivalent to Ryūō Nim.
If $p = 4$, it will be a movement like adding a movement of Knight to Ryuo.

Restrict the diagonal movement version

The Grundy value of this game position turned out to be as follows:

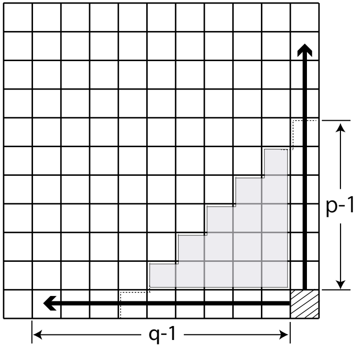
Theorem

Let (x, y) be a Generalized Ryūō Nim position, then we have

$$\mathcal{G}(x, y) = ((x + y) \bmod p) + p(\lfloor \tfrac{x}{p} \rfloor \oplus \lfloor \tfrac{y}{p} \rfloor)\ (p \in \mathbb{Z}_{>1}).$$

Generalized Ryūō Nim

Restrict the diagonal movement by $p \in \mathbb{Z}_{>1}$ and side movement by $q \in \mathbb{Z}_{>1}$.
(It is possible to take up to a total of p tokens when taking them at once and up to q tokens when taking them from one heaps.)



Restrict the diagonal and side movement version

In this case, Grundy value is known only in the following cases:

Theorem

If $q \equiv 0 \pmod{p}$, then we have

$$\mathcal{G}(x,y) = ((x \bmod q + y \bmod q) \bmod p) + p(\lfloor \tfrac{x \bmod q}{p} \rfloor \oplus \lfloor \tfrac{y \bmod q}{p} \rfloor)$$

Restrict the diagonal and side movement version

Theorem

If $q \equiv 1 \pmod{p}$, then we have

(1) $x \equiv 0 \pmod{q}$, $y \equiv 0 \pmod{q}$, $x \neq 0$, $y \neq 0$,

$$\mathcal{G}(x, y) = q$$

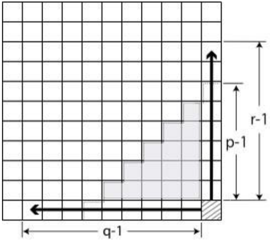
(2) Otherwise

$$\mathcal{G}(x, y) = ((x \bmod q + y \bmod q) \bmod p) + p\left(\left\lfloor \frac{x \bmod q}{p} \right\rfloor \oplus \left\lfloor \frac{y \bmod q}{p} \right\rfloor\right)$$

In other case, it becomes complicated and generally difficult.

Restrict the diagonal and side movement version

Restrict the diagonal movement by $p \in \mathbb{Z}_{>1}$, the horizontal movement by $q \in \mathbb{Z}_{>1}$ and the vertical movement by $r \in \mathbb{Z}_{>1}$.



Theorem

If $q \equiv 0 \pmod{p}$ and $r \equiv 0 \pmod{p}$, then we have

$$\mathcal{G}(x, y) = ((x \bmod q + y \bmod r) \bmod p) + p(\lfloor \frac{x \bmod q}{p} \rfloor \oplus \lfloor \frac{y \bmod r}{p} \rfloor)$$

3-dimensional Ryūō Nim

3-dimensional Ryūō Nim is an impartial game with three heaps of tokens. The rules are as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
- remove one token from any two heaps or
- remove one token from all the three heaps.

The end position is the state of no tokens in the three heaps.

The $\mathcal{P}$-positions of 3-dimensional Ryūō Nim

We could not get the indication of Grundy value for 3-dimensional Ryūō Nim but we get the $\mathcal{P}$-positions as shown in this theorem.

Theorem

Let (x, y, z) be a 3-dimensional Ryūō Nim position.
The $\mathcal{P}$-positions of 3-dimensional Ryūō Nim are given as follows:
$(x + y + z) \equiv 0 \pmod 3$, and moreover

(A) If $x \equiv y \equiv z \equiv 1 \pmod 3$, then

$$\left\lfloor \frac{x}{3} \right\rfloor \oplus \left\lfloor \frac{y}{3} \right\rfloor \oplus \left\lfloor \frac{z}{3} \right\rfloor \oplus 1 = 0$$

(B) Otherwise

$$\left\lfloor \frac{x}{3} \right\rfloor \oplus \left\lfloor \frac{y}{3} \right\rfloor \oplus \left\lfloor \frac{z}{3} \right\rfloor = 0.$$

3-dimensional Ryūō Nim

・Let's change the rule of 3-dimensional Ryūō Nim as follows:
(We will eliminate the rule of taking tokens one by one from the three heaps.)

3-dimensional Ryūō Nim is an impartial game with three heaps of tokens. The rules are as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
- remove one token from any two heaps or
- ~~remove one token from all the three heaps~~.

The end position is the state of no tokens in the three heaps.

The Grundy value of 3-dimensional Ryūō Nim

Then, we can obtain the Grundy value of this game as follows:

Theorem

Let (x, y, z) be a 3-dimensional Ryūō Nim position, then we have

$$\mathcal{G}(x, y, z) = ((x + y + z) \bmod 3) + 3(\lfloor \tfrac{x}{3} \rfloor \oplus \lfloor \tfrac{y}{3} \rfloor \oplus \lfloor \tfrac{z}{3} \rfloor).$$

We considered that this could be expanded and made the following conjecture:

Conjecture

Let $(x_1, \ldots, x_n)$ be a n-dimensional Ryūō Nim position, then we have

$$\mathcal{G}(x_1, \ldots, x_n) = ((x_1 + \cdots + x_n) \bmod 3) + 3(\lfloor \tfrac{x_1}{3} \rfloor \oplus \cdots \oplus \lfloor \tfrac{x_n}{3} \rfloor).$$

In the near future, We'd like to consider whether or not it will be expanded.

References

- Wythoff, W. A ., *A Modication of the Game of Nim*, Nieuw Arch, Wisk. 8, 1907/1909.
- J. H. Conway, *On Numbers And Games* (second edition), A. K. Peters, 2001.
- E. R. Berlekamp, J. H. Conway, R. K. Guy, *Winning Ways for Your Mathematical Plays*, Vols. 1–4, A. K. Peters, 2001–2004.
- Michael H. Albert, Richard J. Nowakowski, David Wolfe, *Lessons in Play, An Introduction to Combinatorial Game Theory*, A. K. Peters 2007.
- A. N. Siegel, *Combinatorial Game Theory*, American Mathematical Society, 2013.

Thank you!