Digital plane preimage structure

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Abstract

In digital geometry, digital straightness is an important concept both for practical motivations and theoretical interests. Concerning the digital straightness in dimension 2, many digital straight line characterizations exist and the digital straight segment preimage is well known. In this article, we investigate the preimage associated to digital planes. More precisely, we present structure theorems that describe the preimage of a digital plane. Furthermore, we present a bound on the number of preimage faces.

Key words: digital plane preimage, digital straight line, dual transformation.

1 Introduction

Digital straightness is an important concept in computer vision. In dimension two, for nearly half a century many digital straight line characterizations have been proposed with interactions with many fields such as arithmetic or theory of words (refer to [1] for a survey on digital straight line). A classical way to define a digital straight line is to consider the digitization of an Euclidean straight line on a unit grid. Hence, given a finite subset of a digital straight line, called digital segment, we can characterize the set of Euclidean straight lines whose digitization contains the digital straight segment. Many authors have discussed about this set of straight lines, also called *preimage*, of a digital segment [2–4]. An important result is that such a preimage is a convex polygon in the parameter space and this domain has got an important arithmetical structure that limits to four the number of vertices. The interest of such a result is double: on one hand we have a better understanding of this simple digital object and on the other hand, we can design very efficient digital straight line recognition algorithms. Concerning digital plane, some algorithms exist in order to decide if a set of grid points in dimension three is a part of a digital plane [5–9]. However, no results have been proposed concerning the structure of the digital plane preimage. In this article, we present several results that describe faces and vertices of the preimage polyhedron in the parameter space.

In section 2, we present major results on the digital straight line preimages. The structure theorems for straight lines are then used to characterize digital plane preimage in section 3. Finally, we present in section 4 a bound on the number of faces of the digital plane preimage.

2 Digital straight line preimage

In the following, we use the notations proposed by Lindenbaum and Bruckstein [4]. Consider a straight line $y = \alpha_0 x + \beta_0$ (with $0 \leq \alpha_0, \beta_0 \leq 1$), the digitization of this line using the Object Boundary Quantization (see [10] for a survey on digitization scheme) on an $N \times N$ unit grid is the set of discrete points such that $L_0 = \{(x, y) \in N \times N \mid \lfloor \alpha_0 x + \beta_0 - y \rfloor = 0\}$. The preimage of a digital straight segment (DSS for short) L_0 is defined by the set of straight lines whose digitization contains L_0. The preimage of L_0, denoted $D(L_0)$, is the set of (α, β) in the straight line parameter space satisfying:

$$D(L_0) = \{(\alpha, \beta) \mid \forall (x, y) \in L_0,\ y \leq \alpha x + \beta < y + 1\} \tag{1}$$

The preimage of a digital straight line on an $N \times N$ grid is given by intersection of linear inequations in the parameter space. Many works have been done concerning the preimage analysis. In the following, we recall properties presented by Dorst and Smeulders [2], McIlroy [3] and Lindenbaum and Bruckstein [4].

Proposition 1 *The domain $D(L_0)$ is a convex polygon in the parameter space with at most four vertices. If $D(L_0)$ has four vertices, two of them have the same α coordinate which is between the α coordinates of the other two vertices.*

The figure 1 illustrates all the possible shapes of $D(L_0)$ (see [4]) and a simple proof can be found in [3]. Among all the various definitions of DSS, we retain the one proposed by Reveillès [11] and based on the following definition:

Definition 2 *An arithmetical naive straight line on an $N \times N$ grid, denoted $N(a, b, \mu)$, with $a, b, \mu \in \mathbb{Z}$ and $gcd(a, b) = 1$ is defined by the set of pixels* *satisfying:*

$$N(a, b, \mu) = \{(x, y) \in N \times N \mid \mu \leq ax - by < \mu + \max(|a|, |b|)\} \quad (2)$$

a/b is the slope of the digital line and μ is the lower bound.

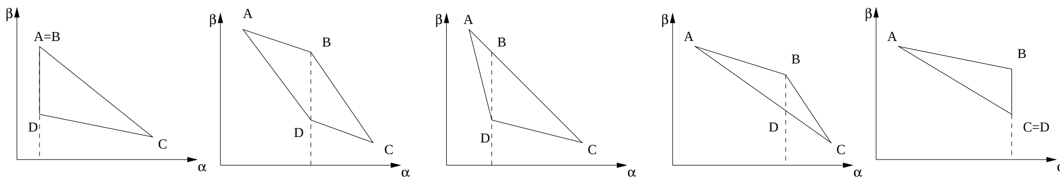


Fig. 1. Five possible shapes of the preimage $D(L_0)$ of a digital straight segment.

If we consider a naive straight line such that $0 \leq a < b$ and thus $\max(|a|, |b|) = b$, we have an equivalence between this characterization and the previous one:

Theorem 3 (Reveillès [11]) *For all α_0 and β_0 such that $0 \leq \alpha_0, \beta_0 \leq 1$, there exist a, b, $\mu \in \mathbb{Z}$ with $0 \leq a < b$ such that $L_0 = N(a, b, \mu)$.*

We choose the Reveillès digital straight line representation scheme because it allows simple illustration of the geometry in the primal space of preimage vertices. More precisely, we can define characteristic points, called *leaning points*, defined as follows: *upper leaning points* (resp. *lower leaning points*) of a digital straight line $N(a, b, \mu)$ are grid points (x, y) satisfying $ax - by = \mu$ (resp. $ax - by = \mu + \max(|a|, |b|) - 1$). We denote U (resp. U') the upper (resp. the lower) leaning point of $N(a, b, \mu)$ with minimum x coordinate (resp. maximum x coordinate). In a same way, we define L and L' from lower leaning points, the figure 2-(a) illustrates these definitions. Using these arithmetical

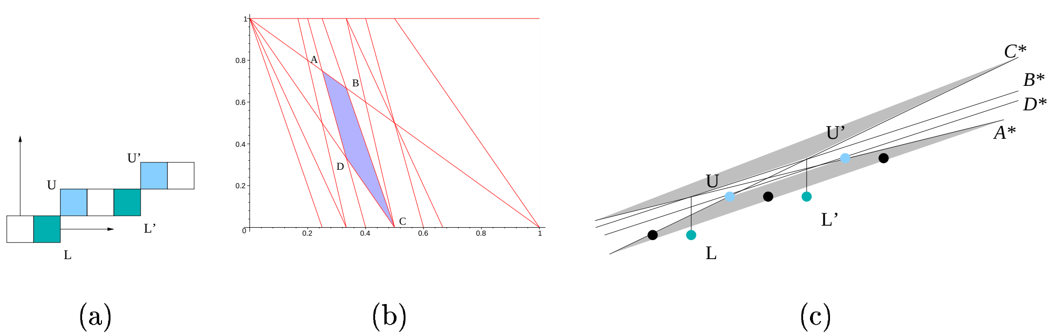


Fig. 2. Illustration in the primal space of the preimage vertices using arithmetical digital line formalism: (a) an arithmetical digital straight line $N(1, 3, 1)$ with lower and upper leaning points, (b) its associated preimage and (c) illustration in the primal space of the preimage vertices.

digital lines, the preimage vertices can be expressed using U, U', L, L' (see figure 2):

- the vertex D corresponds to the straight line (UU') in the primal space ;
- the vertex B corresponds to the straight line $(L_* L'_*)$ where L_* (resp. L'_*) is L (resp. L') translated by the vector $(0,1)^T$;
- the vertex A corresponds to the straight line $(L_* U')$;
- the vertex C corresponds to the straight line $(L'_* U)$.
- the coordinates of D and B are respectively $(\frac{a}{b}, \frac{\mu}{b})$ and $(\frac{a}{b}, \frac{\mu+1}{b})$.

If the preimage has only three vertices, similar results can be derived. In a digital line recognition point of view, the following results can be derived from [11] and [4]:

Theorem 4 *Let $N(a,b,\mu)$ be a digital naive line on an $N \times N$ grid, and a pixel p at the left (or right) side of $N(a,b,\mu)$ and such that p belongs to this straight line. The preimage of the digital line on an $(N+1) \times (N+1)$ grid remains unchanged if and only if p is not a leaning point of $N(a,b,\mu)$.*

3 Digital plane preimage

3.1 Notations and definitions

We consider now a cubic unit grid of size N. The digitization P_0 of an Euclidean plane given by the normal vector $(\alpha_0, \beta_0, \gamma_0) \in [0,1]^2 \times [0,1[$ is the set of grid points (called *voxels* in 3D) satisfying:

$$P_0 = \Big\{(x,y,z) \in N^3 \mid \lfloor \alpha_0 x + \beta_0 y + \gamma_0 - z \rfloor = 0\Big\} \quad (3)$$

In the same manner as in 2D, we can define the preimage of the digital plane P_0 considering the set of parameters (α, β, γ) such that the digitization of the associated plane is P_0:

$$D_{3D}(P_0) = \{(\alpha,\beta,\gamma) \in [0,1]^2 \times [0,1[\mid \forall (x,y,z) \in P_0,\ z \leq \alpha x + \beta y + \gamma < z+1\} \quad (4)$$

The preimage, denoted $D_{3D}(P_0)$, is a convex polyhedron in the (α,β,γ) parameter space because it is the intersection of linear inequalities. We also consider a characterization of the digital plane based on *arithmetical naive plane* [11]:

Definition 5 *An arithmetical naive plane in a N^3 grid, denoted $P(a,b,c,\mu)$, with $a,b,c,\mu \in \mathbb{Z}$ and $gcd(a,b,c) = 1$ is defined by the set of voxels satisfying:*

$$P(a,b,c,\mu) = \{(x,y,z) \in N^3 \mid \mu \leq ax + by + cz < \mu + \max(|a|,|b|,|c|)\} \quad (5)$$

$(a, b, c)^T$ *is the digital plane normal vector and* μ *is the lower bound.*

In the following, we consider naive plane such that $0 \leq a \leq b < c$ and thus $\max(|a|, |b|, |c|) = c$. The digital plane characterization is the same as the definition given by equation 3: for each plane P_0 given by $(\alpha_0, \beta_0, \gamma_0)$, there exist $a, b, c, \mu \in \mathbb{Z}$ with $0 \leq a \leq b < c$ such that $P_0 = P(a, b, c, \mu)$. In this arithmetical plane, we can also define special voxels, so called *upper and lower leaning points*: the upper leaning points are voxels satisfying $ax + by + cz = \mu$ and lower leaning points are voxels satisfying $ax + by + cz = \mu + \max(|a|, |b|, |c|) - 1$. Since these points are respectively coplanar, we also define the *upper leaning polyon*, denoted L_{up} (resp. *lower leaning polygon* denoted L_{low}) by the 2D convex hull of upper leaning points (resp. lower leaning points). The figure 3 illustrates these definitions.

In the next section, we present links between preimage faces and leaning polygon vertices.

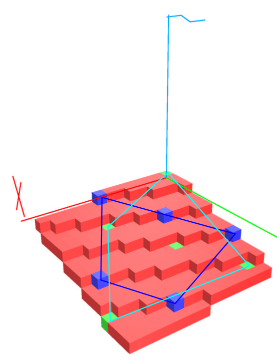


Fig. 3. Illustration of an arithmetical digital plane $P(7, 17, 57, 0)$, lower and upper leaning points and lower and upper leaning polygons with $N = 15$.

3.2 Digital plane preimage characterization

We first introduce vertices and faces of $D_{3D}(P)$ given by leaning polygons:

Proposition 6 *Let* $P(a, b, c, \mu)$ *be a piece of naive plane. Then, the polyhedron containing all the Euclidian planes* $D_{3D}(P)$ *in the parameter space has the following properties :*

- *Two particular vertices with coordinates* $L^*_{\text{low}}(\frac{a}{c}, \frac{b}{c}, \frac{\mu}{c})$ *and* $L^*_{\text{up}}(\frac{a}{c}, \frac{b}{c}, \frac{\mu+1}{c})$ *which corresponds to the planes containing the leaning polygons* L_{up} *and* L_{low} *in the primal space;*
- *the polyhedron's faces adjacent to* L^*_{low} *(resp.* L^*_{up}*) result from the lower (resp. upper) leaning polygon's vertices.*

PROOF. In the (α, β, γ) parameter space, each point $p(x, y, z)$ in P introduces two linear constraints $C_1(p): \ \alpha x + \beta y + \gamma - z \geq 0$ and $C_2(p): \ \alpha x + \beta y + \gamma - z - 1 < 0$ with $(\alpha, \beta, \gamma) \in [0,1]^2 \times [0,1[$. Since (α, β, γ) are positive and according to Preparata and Shamos [12], the domain $D_{3D}(P)$ is given by computing the lower envelope of constraints C_2, by computing the upper envelope of constraints C_1 and by merging these two envelopes. In other words, we can treat separately constraints C_1 and C_2.

Hence, we consider the constraints $\{C_2\}$ of points p_i in P and the leaning plane containing the upper leaning polygon L_{up}. Since all points p_i are below the leaning plane by definition of this plane, all half-planes defined by the constraints $\{C_2(p_i)\}$ contain the point L^*_{up} in the parameter space. Thus, since all upper leaning points have constraints C_2 going through L^*_{up}, L^*_{up} is necessarily a vertex of the lower envelope of constraints $\{C_2\}$ and so, L^*_{up} is necessarily a vertex of the polyhedron $D_{3D}(P)$. Using same arguments, we prove that L^*_{low} is a vertex, in the parameter space, of the upper envelope of constraints $\{C_1(p_i)\}$ and thus, L^*_{low} is also a vertex of $D_{3D}(P)$. Coordinates of L^*_{up} and L^*_{low} are given by definition of leaning points.

If we consider now the adjacent faces to the point L^*_{up} of $D_{3D}(P)$, each face with normal vector $(x_i, y_i, z_i)^T$ is created by the upper leaning point with coordinates (x_i, y_i, z_i). We denote $\{e^i\}_{1\ldots m}$ the vertices of the leaning polygon L_{up} and v a coplanar voxel to points $\{e^i\}_{1\ldots m}$, inside the polygon. Since L_{up} is the planar convex hull of upper leaning points, we have: $v = \sum_{i=1}^{m} \omega_i e^i$, where $\{\omega_i\}_{1\ldots m}$ are vectors in $\mathbb{R}^3$ with positive coordinates. Then, the constraint generated by v in the dual space contains L^*_{up} and has a normal vector which is linearly dependent with positive weights to normal vectors of faces $\{e^i\}_{1\ldots m}$ (see figure 4). Thus, v is not an adjacent face to L^*_{up} in $D_{3D}(P)$. Finally, all the adjacent faces to L^*_{up} are only generated by the upper leaning polygon's vertices. Similarly, all the adjacent faces to L^*_{low} in the parameter space are generated by lower leaning polygon's vertices. □

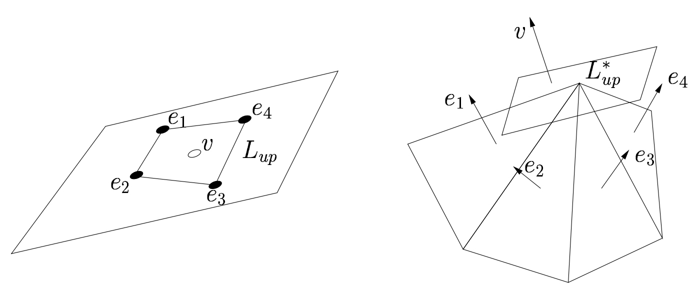


Fig. 4. Illustration of proposition 6: $(left)$ vertices $\{e^i\}_{1\ldots m}$ of the upper leaning polygon and the point v lying inside this polygon, $(right)$ the constraint generated by v has a normal verctor linearly dependent with positive weights to normal vectors of faces $\{e^i\}_{1\ldots m}$ in the parameter space.

At this point, we have proved that the preimage $D_{3D}(P)$ has two characteristic vertices associated to the leaning planes and particular faces created by the leaning polygon's vertices. In the following, we prove that, with some hypothesis on the digital plane, the preimage $D_{3D}(P)$ does not contain other faces.

Definition 7 *Let $P(a,b,c,\mu)$ (with $0 \leq a \leq b < c$) be a piece of naive plane. We define the* **double-cone** *in the parameter space associated to P and denoted by $D_{cone}(P)$ the domain where faces are generated by leaning polygons' vertices and with two characteristic points generated by leaning planes.*

The two following theorems show that we have $D_{3D}(P) = D_{cone}(P)$ with some hypothesis on P.

Theorem 8 *Let $P(a,b,c,\mu)$ (with $0 \leq a \leq b < c$) be a piece of naive plane where each point (x_i, y_i, z_i) is such that (x_i, y_i) lies inside the projections onto the plane $z = 0$ of both leaning polygons. Then we have $D_{3D}(P) = D_{cone}(P)$.*

PROOF. Let consider a voxel v that belongs to the digital plane $P(a,b,c,\mu)$ and that satisfies theorem hypothesis. Let us first consider the constraint $C_2(v)$ and we show that $C_2(v)$ does not intersect $D_{cone}(P)$.

Since v belongs to P, $C_2(v)$ necessarily contains the point $L^*_{\mathrm{up}}(\frac{a}{c}, \frac{b}{c}, \frac{\mu+1}{c})$ in the parameter space. In other words, the plane $C_2(v)$ crosses the straight lines $(L^*_{\mathrm{low}} L^*_{\mathrm{up}})$ at a point p with γ coordinate greater than the γ coordinate of L^*_{up} (see figure 5). If $C_2(v)$ crosses the domain $D_{cone}(P)$, then the translation of $C_2(v)$ by the vector $\overrightarrow{pL^*_{\mathrm{up}}}$ crosses the domain. This transformation translates the plane $C_2(v)$ into a plane C' that goes through L^*_{up}. In the primal space, this translation corresponds to a vertical projection of the voxel v onto the upper leaning plane. According to the hypothesis on the digital plane voxels, this vertical projection of v lies inside the upper leaning polygon. Thus, using the same arguments as in the proof of proposition 6, the normal vector of C' is linearly dependent with positive weights of the face normal vectors created by upper leaning polygons' vertices. Hence, C' does not belong to the lower envelope of constraints $\{C_2\}$ and does not cross the domain $D_{cone}(P)$. Then, $C_2(v)$ does not cross the domain too.

Considering the constraint $C_1(v)$, similar arguments are used with projection onto the lower leaning plane. Finally, if all voxels of P are such that the vertical projection of such points lies inside both leaning polygon projections, the voxel v does not change the preimage and thus: $D_{3D}(P) = D_{cone}(P)$. □

In the following we prove that for a digital plane containing at least three leaning points on each line along the y axis or the x axis, $D_{\mathrm{3D}}(P)$ does not

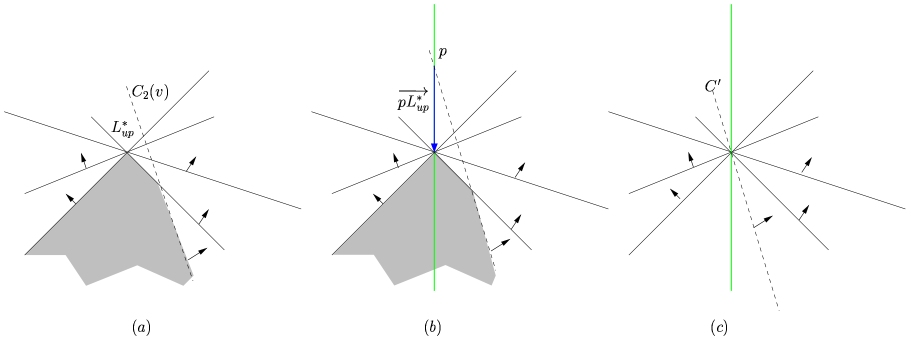


Fig. 5. Illustration of the proof of theorem 8 in the 2D case.

contain more faces than those described in proposition 6. In order to prove this statement, we use the theorem 4 on digital line preimage presented in section 2 and the following decomposition of a digital plane into digital lines.

Proposition 9 *Let $P(a, b, c, \mu)$ be a naive plane. Let us define the decomposition of P into 3D digital straight lines along the y axis : $S_{\mathrm{j}} = \{(x, y, z) \in P \mid y = j\}$. Then we have:*

$$P = \bigcup_{j} S_{\mathrm{j}} \quad \textit{and} \quad D_{\mathrm{3D}}(P) = \bigcap_{\mathrm{j}} D_{\mathrm{3D}}(S_{\mathrm{j}}) \tag{6}$$

If we denote $P(a, b, c, \mu)$ the digital naive plane, we can map each set of voxels S_{j} to a digital naive line in the (Oxy) plane. This map is one to one and onto, and the digital line is exactly $N(a, c, \mu + bj)$. In the parameter space, the preimage of S_{j} is a prism such that the basis (for $\beta = 0$ and $\beta = 1$) is the preimage of $N(a, c, \mu + bj)$ and such that the directional vector is $(0, 1, -j)^T$. Figure 6 shows an example of a 3D line $\{S_{\mathrm{j}}\}$ preimage and figure 7 illustrates the digital plane preimage computation based on the $\{S_{\mathrm{j}}\}$ preimage intersections. Note that the digital plane leaning points are also the leaning points for $N(a, c, \mu + bj)$.

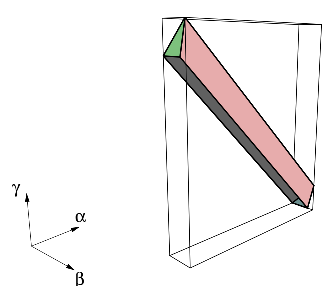


Fig. 6. Preimage of the 3D digital straight line defined by $y = 1$ in the plane $P(1, 3, 4, 0)$

From this decomposition we can derive the following theorem :

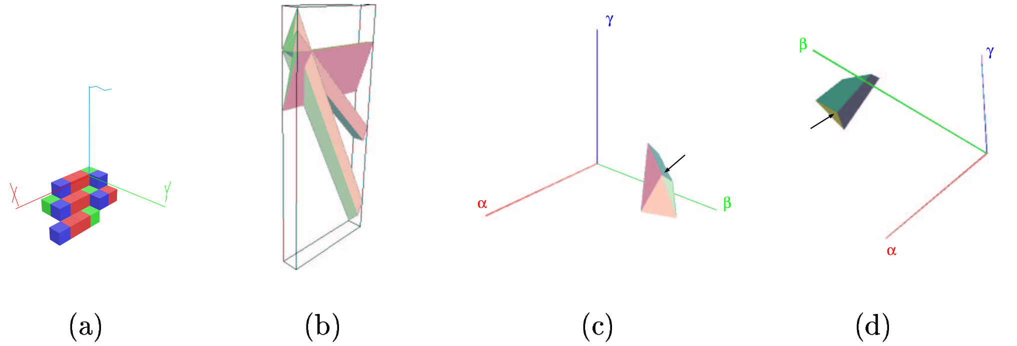


Fig. 7. (a) A piece of plane $P(1,3,4,0)$; (b) preimages of 3D digital straight lines in parameter space (α,β,γ); (c)-(d) the preimage of the piece of plane is the intersection of the digital lines preimages, arrows aim at L^*_{up} (figure (c)) and L^*_{low} (figure (d)).

Theorem 10 *Let $P(a,b,c,\mu)$ a piece of discrete naive plane. Let $P = \bigcup_j S_{\text{j}}$ with $S_{\text{j}} = \{(x,y,z) \in P \mid y = j\}$. Then, if each $\{S_{\text{j}}\}$ contains at least three leaning points, we have $D_{\text{3D}}(P) = D_{\text{cone}}(P)$.*

PROOF. Let us consider the decomposition of P into 3D digital lines $\{S_{\text{j}}\}$. Since, for all j, S_{j} contains at least 3 leaning points, according to what we saw, the preimage of $\{S_{\text{j}}\}$ is based on the preimage of the digital straight line $N(a,c,\mu+bj)$. Let us consider a voxel $v(x,y,z)$ which is not a leaning point of P, this voxel belongs to one and only one line of the decomposition, S_{y}. Now let us consider the projection of this line in the 2D space (Oxz), and denote it $Proj(S_{\text{y}})$. As S_{y} contains at least three leaning points, the digital line $Proj(S_{\text{y}})$ also have at least three leaning points. Furthermore, if v is not a leaning point for the digital plane, v is not a leaning point for $Proj(S_{\text{y}})$. According to theorem 4, preimage of $Proj(S_{\text{y}})$ does not change after the insertion of the voxel v and thus the preimage of S_{y} does not change too. This means that the constraints resulting from the voxel v contain entirely the preimage of S_{y}.

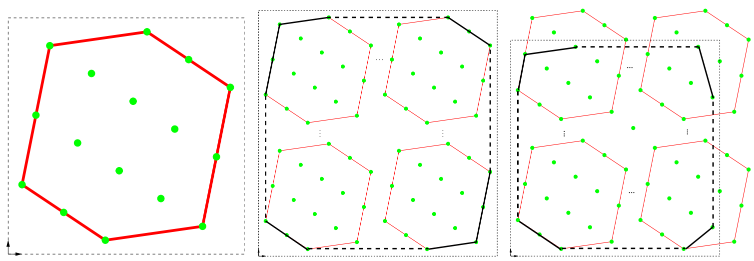
Finally, since $D_{\text{3D}}(P)$ is the intersection of all $\{S_{\text{j}}\}$ preimages and since v does not modify the S_{y} prism, v does not change the domain $D_{\text{3D}}(P)$. Then, we obtain the expected result. □

4 Bounds on the number of faces

It has been proved in previous theorems how to construct the preimage of a digital plane. In this section, we present a bound on the number of faces of this preimage. Let us suppose a digital plane $P(a,b,c,\mu)$ satisfying hypothesis of theorems either 8 or 10 (or both). The number of faces of D_{3D} is exactly the number of both leaning polygons' vertices.

As given in definition 5, an arithmetical plane $P(a, b, c, \mu)$ is composed of a set of arithmetical nets given by the solution of the diophantine equation $ax + by + cz = r$ with r in the interval $[\mu, \mu + \max(|a|, |b|, |c|)[$. Given a digital plane on a unit grid of size N, the problem is to bound the number of vertices of upper (resp. lower) leaning point convex hull. First note that the upper (resp. lower) leaning net can be projected onto the $(0xy)$ plane without changing the number of vertices of the convex hull. The problem is to consider the convex hull size of the bidimensional net $ax + by = r$ in a $N \times N$ window. We first construct two vectors, denoted $U(p, q)$ and $V(s, t)$, that compose a basis of the net using the classical Blankinship's algorithm in number theory [13]. In other words, all upper leaning points are generated by these two vectors. Using scale changes on the grid axis, we can construct a net defined by canonical vectors $(1, g)$ and $(1, h-g)$. This one to one and onto mapping from the net generated by $[U, V]$ to the net generated by $[(1, g), (1, h-g)]$ does not change the number of convex hull vertices (given two vectors in the plane, the transformation does not change the sign of the determinant of those vectors).

The net generated by $[(1, g), (1, h-g)]$ in an $h \times h$ window (see figure 8-(a)), is exactly the net $\{(i, gi \bmod h)\}$ with $0 < i < h$. As proved by Reveillès and Yaacoub in [14], the number of vertices of the convex hull of such points is in $\mathcal{O}(\log(g))$ (authors illustrate links between such a net and continued fraction of g/h). Over a square $[0, h[\times[0, h[$, the complexity is known. On this specific window, we can define the farthest point in the north, south, east and west direction. Since we have a general window $[0, m[\times[0, n[$ whose size depends on the size of the digital plane under study, we first consider the square $[0, k.h[\times[0, l.h[$ (with $k, l \in \mathbb{Z}$). In each period the position of the north point is the same, thus upper part of the convex hull in the square $[0, k.h[\times[0, l.h[$ is a horizontal segment between the north points of the periods $[0, k.h[\times[(l-1).h, l.h[$. In a same way, the lower part of the convex hull is ahorizontal segment defined by south points, the left and right) part are vertical segments respectively defined by west points and east points. Hence, on a $[0, k.h[\times[0, l.h[$ window, the convex hull can be decomposed into four horizontal and vertical segments and four parts of convex hull on $[0, h[\times[0, h[$ (see figure 8-(c)). The size of such a convex hull does not depend on k or l. As a consequence, we can reduce the study of the complexity to the complexity of the convex hull over the square $[0, 2.h[\times[0, 2.h[$. In such a square, the complexity is thus bounded by four times the complexity of the convex hull over one period. With the result of Reveillès and Yaacoub, we deduce that the complexity is still $\mathcal{O}(\log(g))$. We consider now the square $[0, m[\times[0, n[$. We can defined the farthest point in the north, south, east and west directions as previously. Using the same argument we reduce to a sub-square of $[0, 2.h[\times[0, 2.h[$ that is $[0, h + m \bmod h[\times[0, h + n \bmod h[$ (see Figure 8-(c)). On such a window, we can decompose the convex hull into four parts corresponding to the four periods of the net. The convex hull is thus the connection of these four parts by four edges. Hence, the size of the convex hull is bounded by four times the size of the polygonal arcs in the four periods. However, each part of the convex hull is a convex polygonal curve inscribed inside the convex hull of one period of the net. Thus, its length in number of edges if bounded above by the length of the border of the convex hull of the points of one period. We know this bound to be $\log(g)$ and thus the complexity of the complete convex hull is still in $\log(g)$.



(a) (b) (c)

Fig. 8. (a) set of points generated by $\{(i, gi \bmod h)\}$ with $g = 5$ and $h = 17$; (b) convex hull computation on a $[0, k.h[\times[0, l.h[$ window; (c) convex hull computation on a $[0, m[\times[0, n[$ window.

Theorem 11 *Let P be a digital plane on an N^3 unit grid satisfying hypothesis of theorems either 8 or 10 (or both). Then the number of faces of the preimage of P is bounded by $\mathcal{O}(log(N))$.*

5 Conclusion

In this paper we have presented some results about digital plane preimage. We have shown that with some hypothesis on the piece of digital plane, the shape of the preimage is a double-cone which structure is very similar to the one of 2D digital straight segments preimages. Nevertheless, we conjecture that this property is true for any digital plane segment without any assumption.

We have also introduced the decomposition of a digital plane segment into 3D digital straight segments, which suggests interesting arithmetical properties on the polyhedron's faces and vertices. Indeed, each 3D digital line segment preimage face is resulting from a side of a 2D digital segment preimage which have known arithmetical structure.

Finally, we have shown that under some hypotheses, the number of faces of a digital plane segment preimage is bounded by $\mathcal{O}(log(N))$ if the piece of plane is in a N^3 grid. As in 2D, such a result together with the other ones of this paper can lead to the design of a very efficient digital plane recognition algorithm.

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