Octal Games on Graphs

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This work is part of the ANR GAG (Graphs and Games).

CGTC 2017
Octal games

Definition

Octal games are:

- impartial games;
- played on heaps of counters;
- whose rules are defined by an octal code.

Examples

- Nim is 0.3333...
- Kayles is 0.137
- Cram on a single row is 0.07
- The James Bond Game is 0.007
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Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size 0, 1, 2, ...
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Examples

- **NIM**: 0, 1, 2, 3, 4, 5, . . .
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Examples

- **NIM**: 0, 1, 2, 3, 4, 5, ...
- **KAYLES**: 0, 1, 2, 3, 1, 4, 3, 2, ... after a **pre-period 72** it becomes periodic with **period 12**;
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- **CRAM** on a single row: 0, 1, 1, 2, 0, 3, 1, 1, . . . after a pre-period 53 it becomes periodic with period 34
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- **The James Bond Game**: 0, 0, 0, 1, 1, 1, 2, 2, 0, 3, 3, 1, 1, 1, 0, 4, \ldots still open, $2^{28}$ values computed!
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Conjecture (Guy)

All **finite** octal games have **ultimately periodic** Grundy sequences.
Octal games on graphs

Natural generalization of the definition:

<table>
<thead>
<tr>
<th>Playing on heaps</th>
<th>Playing on graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Heap Diagram]</td>
<td>![Graph Diagram]</td>
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Removing counters from a heap
Disconnecting a graph

Playing on a heap ≡ Playing on a path
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- Removing counters from a heap
- Disconnecting a graph
- Splitting a heap
- Removing connected vertices from a graph
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```text
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\text{Removing *counters* from a *heap*} \\
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\text{Playing on graphs} \\
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\end{array}
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I I I I
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![Graph Diagram](image)
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```
I   I   I   I
```

```
\begin{tikzpicture}
    \draw (0,0) -- (0.5,0);
    \draw (0,0) -- (0,0.5);
    \draw (0,0) -- (0,-0.5);
\end{tikzpicture}
```
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  \item \textbf{Splitting} a heap
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Octal games on graphs

Natural generalization of the definition:

Playing on heaps

Removing **counters** from a **heap**

Splitting a heap

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```
   1 1 1 1
```

```
    o---o---o---o
    |             |
    |             |
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[Diagram of a graph with disconnected vertices]
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Playing on a heap $\equiv$ Playing on a path
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- **grim** (Adams et al., 2016) is 0.6.

- Study of cycles, wheels, random graphs, ... Scoring version of 0.6 (Duchêne et al., 2017+).

- **node-kayles** is not an octal game.
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The game 0.33 on graphs

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In the game 0.33, both players alternate removing one or two adjacent vertices without disconnecting the graph.
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Remark
For every integer $n$, we have $G(P_n) = n \mod 3$. 
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Remark
For every integer $n$, we have $G(P_n) = n \mod 3$.

Corollary
A path can be reduced to its length modulo 3 without changing its Grundy value.
The game 0.33 on subdivided stars

Subdivided stars
A subdivided star $S_{\ell_1, \ldots, \ell_k}$ is a graph composed of a central vertex connected to $k$ paths of length $\ell_1, \ldots, \ell_k$.

\[
\begin{align*}
S_{1,1,2} & \quad S_4 & \quad S_{1,2,3,6} & \quad S_{1,1,1,1,1,1,1,1}
\end{align*}
\]
The game 0.33 on subdivided stars

Subdivided stars
A subdivided star $S_{\ell_1, \ldots, \ell_k}$ is a graph composed of a central vertex connected to $k$ paths of length $\ell_1, \ldots, \ell_k$.

\begin{center}
\begin{tabular}{ccc}
$S_{1,1,2}$ & $S_4$ & $S_{1,2,3,6}$ & $S_{1,1,1,1,1,1,1,1}$
\end{tabular}
\end{center}

Theorem
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$. In other words, each path of a subdivided star can be reduced to its length modulo 3 without changing the Grundy value of the star.

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$S_{1,2,3,6}$
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Subdivided stars
A subdivided star \( S_{\ell_1, \ldots, \ell_k} \) is a graph composed of a central vertex connected to \( k \) paths of length \( \ell_1, \ldots, \ell_k \).

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\begin{align*}
S_{1,2,3,6} & \equiv S_{1,2} = P_4
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Subdivided stars

A **subdivided star** $S_{\ell_1, \ldots, \ell_k}$ is a graph composed of a central vertex connected to $k$ paths of length $\ell_1, \ldots, \ell_k$.

![Graph diagrams](image-url)

$S_{1,1,2}$  $S_4$  $S_{1,2,3,6}$  $S_{1,1,1,1,1,1,1,1,1}$

**Theorem**

For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

In other words, each path of a subdivided star can be **reduced** to its **length modulo 3** without changing the Grundy value of the star.

![Graph diagrams](image-url)

$S_{1,2,3,6}$  $S_{1,2} = P_4$  $P_1$
The game 0.33 on subdivided stars

Theorem
For all \( \ell_1, \ldots, \ell_k \), we have
\[
G(S_{\ell_1}, \ldots, \ell_k) = G(S_{\ell_1 \mod 3}, \ldots, \ell_k \mod 3).
\]
The game 0.33 on subdivided stars

Theorem
For all \( \ell_1, \ldots, \ell_k \), we have \( G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3}) \).

Proof
We prove by induction that \( G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i + 3, \ldots, \ell_k}) \).
The game 0.33 on subdivided stars

Theorem
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

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We prove by induction that $G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i+3, \ldots, \ell_k})$. 

\[ + \]
The game 0.33 on subdivided stars

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For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

Proof
We prove by induction that $G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i+3, \ldots, \ell_k})$. 

\[\begin{align*}
\text{Diagram}
\end{align*}\]
The game 0.33 on subdivided stars

**Theorem**
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

**Proof**
We prove by induction that $G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i + 3, \ldots, \ell_k})$. 
The game 0.33 on subdivided stars

**Theorem**
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1}, \ldots, \ell_k) = G(S_{\ell_1 \mod 3}, \ldots, \ell_k \mod 3)$.

**Proof**
We prove by induction that $G(S_{\ell_1}, \ldots, \ell_i, \ldots, \ell_k) = G(S_{\ell_1}, \ldots, \ell_i + 3, \ldots, \ell_k)$.
The game 0.33 on subdivided stars

Theorem
For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1, \ldots, \ell_k}) = \mathcal{G}(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

Proof
We prove by induction that $\mathcal{G}(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = \mathcal{G}(S_{\ell_1, \ldots, \ell_i+3, \ldots, \ell_k})$. 

\begin{itemize}
  \item \[ \begin{array}{c}
  \mathcal{G}(S_{\ell_1}) = \mathcal{G}(S_{\ell_1+3}) \\
  \mathcal{G}(S_{\ell_1} + S_{\ell_2}) = \mathcal{G}(S_{\ell_1+3} + S_{\ell_2+3}) \\
  \mathcal{G}(S_{\ell_1} + S_{\ell_2} + S_{\ell_3}) = \mathcal{G}(S_{\ell_1+3} + S_{\ell_2+3} + S_{\ell_3+3}) \\
  \mathcal{G}(S_{\ell_1} + S_{\ell_2} + S_{\ell_3} + S_{\ell_4}) = \mathcal{G}(S_{\ell_1+3} + S_{\ell_2+3} + S_{\ell_3+3} + S_{\ell_4+3}) \\
  \mathcal{G}(S_{\ell_1} + S_{\ell_2} + S_{\ell_3} + S_{\ell_4} + S_{\ell_5}) = \mathcal{G}(S_{\ell_1+3} + S_{\ell_2+3} + S_{\ell_3+3} + S_{\ell_4+3} + S_{\ell_5+3}) \\
  \mathcal{G}(S_{\ell_1} + S_{\ell_2} + S_{\ell_3} + S_{\ell_4} + S_{\ell_5} + S_{\ell_6}) = \mathcal{G}(S_{\ell_1+3} + S_{\ell_2+3} + S_{\ell_3+3} + S_{\ell_4+3} + S_{\ell_5+3} + S_{\ell_6+3}) \\
  \end{array} \]
\end{itemize}
The game 0.33 on subdivided stars

Theorem
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

Proof
We prove by induction that $G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i + 3, \ldots, \ell_k})$.
The game 0.33 on subdivided stars

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Proof
We prove by induction that $G(S_{\ell_1}, \ldots, \ell_i, \ldots, \ell_k) = G(S_{\ell_1}, \ldots, \ell_i + 3, \ldots, \ell_k)$.

\[
\begin{array}{c}
\text{+} \quad \text{+} \\
\end{array}
\begin{array}{c}
\text{-} \\
\end{array}
\begin{array}{c}
\text{+} \quad \text{+} \\
\end{array}
\]
The game 0.33 on subdivided stars

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For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1}, \ldots, \ell_k) = G(S_{\ell_1 \mod 3}, \ldots, \ell_k \mod 3)$.

**Proof**
We prove by induction that $G(S_{\ell_1}, \ldots, \ell_i, \ldots, \ell_k) = G(S_{\ell_1}, \ldots, \ell_i + 3, \ldots, \ell_k)$.

\[ P_\ell + S_{1,1,\ell} \]
The game 0.33 on subdivided stars

Lemma
For all $\ell$, we have $G(S_{1,1,\ell}) = \ell \mod 3$. 

Proof
We use induction on $\ell$. 

$G = \ell + 2 \mod 3$

$G = \ell - 1 \mod 3$

$G = \ell - 2 \mod 3$

$G = \ell \mod 3$
The game 0.33 on subdivided stars

Lemma
For all \( \ell \), we have \( G(S_{1,1,\ell}) = \ell \mod 3 \).

Proof
We use induction on \( \ell \).

\[
G(\bullet) = 0 \quad G(\circlearrowright) = 1
\]
The game 0.33 on subdivided stars

Lemma
For all $\ell$, we have $G(S_{1,1,\ell}) = \ell \mod 3$.

Proof
We use induction on $\ell$. 
The game 0.33 on subdivided stars

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The game 0.33 on subdivided stars

Lemma
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The game 0.33 on subdivided stars

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For all $\ell$, we have $G(S_{1,1,\ell}) = \ell \mod 3$.

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We use induction on $\ell$.

\[
G = \ell \mod 3
\]

\[
G = \ell + 2 \mod 3
\]

\[
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\]

\[
G = \ell - 2 \mod 3
\]
The game 0.33 on subdivided stars

**Theorem**
For all $\ell_1, \ldots, \ell_k$, we have $G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})$.

**Proof**
We prove by induction that $G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i + 3, \ldots, \ell_k})$.

\[
P_{\ell} + S_{1,1,\ell} \quad \Rightarrow \quad G(P_{\ell} + S_{1,1,\ell}) = 0
\]
The game 0.33 on subdivided stars

Theorem
For all \(\ell_1, \ldots, \ell_k\), we have \(G(S_{\ell_1, \ldots, \ell_k}) = G(S_{\ell_1 \mod 3, \ldots, \ell_k \mod 3})\).

Proof
We prove by induction that \(G(S_{\ell_1, \ldots, \ell_i, \ldots, \ell_k}) = G(S_{\ell_1, \ldots, \ell_i + 3, \ldots, \ell_k})\).

\[
P_\ell + S_{1,1,\ell} \\
G(P_\ell + S_{1,1,\ell}) = 0
\]

\(\Rightarrow\) We only need to study stars with paths of length 1 and 2
Grundy values of subdivided stars for the game 0.33
Grundy values of subdivided stars for the game 0.33

Number of paths of length 2 in the subdivided star

Number of paths in the subdivided star

∅ 0 1 2 3 4 5 ⋯ 2p 2p + 1
Grundy values of subdivided stars for the game 0.33

Number of paths of length 2 in the subdivided star

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & \cdots & 2p & 2p+1 \\
\emptyset & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 2p & 2p+1 \\
\end{array}
\]
Grundy values of subdivided stars for the game 0.33

Number of paths of length 2 in the subdivided star

∅ 0 1 2 3 4 5 \ldots 2p 2p + 1

Number of paths in the subdivided star

0 0 1 2 0 1 2 3 1 2

2p 3 1 2 0 3 1 2

2p + 1 1 2 0 3 1 2

(03)∗ 0
(12)∗ 1 2
The game 0.33 on subdivided bistars

Subdivided bistars

The **subdivided bistar** $S_1 \cdot^m \cdot S_2$ is the graph constructed by joining the central vertices of two subdivided stars $S_1$ and $S_2$ by a path of $m$ edges.

![Diagram of subdivided bistars](image)
The game 0.33 on subdivided bistars

Subdivided bistars
The subdivided bistar $S_1 \ast^m S_2$ is the graph constructed by joining the central vertices of two subdivided stars $S_1$ and $S_2$ by a path of $m$ edges.

Theorem
Each path of a subdivided bistar can be reduced to its length modulo 3 without changing the Grundy value of the bistar.
The game 0.33 on subdivided bistars

Subdivided bistars

The **subdivided bistar** $S_1 \cdot^m \cdot S_2$ is the graph constructed by joining the central vertices of two subdivided stars $S_1$ and $S_2$ by a path of $m$ edges.

\[ S_{1,1} \cdot^1 \cdot S_{1,1} \quad S_{1,2} \cdot^2 \cdot \emptyset \quad S_{1,2,3} \cdot^3 \cdot S_{2,4} \]

**Theorem**

Each path of a subdivided bistar can be reduced to its length modulo 3 without changing the Grundy value of the bistar.

\[ S_{1,2,3} \cdot^3 \cdot S_{2,4} \equiv S_{1,2} \cdot^3 \cdot S_{1,2} \]
The game 0.33 on subdivided bistars

Subdivided bistars

The **subdivided bistar** $S_1 \overset{m}{\rightarrow} S_2$ is the graph constructed by joining the central vertices of two subdivided stars $S_1$ and $S_2$ by a path of $m$ edges.

\[
\begin{align*}
S_{1,1} & \overset{1}{\rightarrow} S_{1,1} \\
S_{1,2} & \overset{2}{\rightarrow} \emptyset \\
S_{1,2,3} & \overset{3}{\rightarrow} S_{2,4}
\end{align*}
\]

**Theorem**

Each path of a subdivided bistar can be **reduced** to its **length modulo 3** without changing the Grundy value of the bistar.

\[
\begin{align*}
S_{1,2,3} & \overset{3}{\rightarrow} S_{2,4} \\
S_{1,2} & \overset{3}{\rightarrow} S_{1,2} \\
\emptyset & \overset{3}{\rightarrow} S_{1,1,2,2}
\end{align*}
\]
The game 0.33 on subdivided bistars

We want to \textit{directly} compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.
The game 0.33 on subdivided bistars

We want to directly compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

Playing on a subdivided bistar

Playing independently on the two subdivided stars
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

\[ G(\text{subdivided bistar}) = 0 \quad G(\text{star}) = 0 \quad G(\text{subdivided bistar}) = 1 \quad G(\text{star}) = 0 \]

\[ \Rightarrow \text{Refinement of } \equiv \frac{12}{19} \]
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

Playing on a subdivided bistar

Playing independently on the two subdivided stars

...except at the end!
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

![Diagram of subdivided bistar and its stars](image)

Playing on a subdivided bistar

...except at the end!

\[
G(\text{subdivided bistar}) = 0 \\
G(\text{subdivided star} + \text{linear star}) = 0
\]
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

\[
\begin{align*}
G(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}) & = 0 \\
G(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}) + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & = 0 \\
G(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}) & = 1 \\
G(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}) + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & = 0
\end{align*}
\]

Playing on a subdivided bistar

\[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}\]

...except at the end!

Playing independently on the two subdivided stars
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

![Diagram of a subdivided bistar and two subdivided stars](image)

Playing on a subdivided bistar

...except at the end!

\[
G(\text{subdivided bistar}) = 0 \\
G(\text{subdivided stars}) = 0
\]

\[
G(\text{subdivided bistar}) = 1 \\
G(\text{subdivided stars}) = 0
\]

⇒ **Refinement** of \(\equiv\)
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$J_1 \equiv J_2 \iff \forall X, J_1 + X$ and $J_2 + X$ have the same outcome.
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

\[ J_1 \equiv J_2 \iff \forall X, \ J_1 + X \text{ and } J_2 + X \text{ have the same outcome.} \]

Refinement of $\equiv$

\[ S \sim_1 S' \iff \forall X, \ S\bullet^1 X \text{ and } S'\bullet^1 X \text{ are equivalent.} \]
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$$J_1 \equiv J_2 \iff \forall X, \ J_1 + X \text{ and } J_2 + X \text{ have the same outcome.}$$

Refinement of $\equiv$

$$S \sim_1 S' \iff \forall X, \ S\cdot^1\cdot X \text{ and } S'\cdot^1\cdot X \text{ are equivalent.}$$

\[
\begin{array}{c}
\text{Refinement of } \equiv \\
\Rightarrow \\
\sim_1
\end{array}
\]
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

\[ J_1 \equiv J_2 \iff \forall X, J_1 + X \text{ and } J_2 + X \text{ have the same outcome.} \]

Refinement of $\equiv$

\[ S \sim_1 S' \iff \forall X, S^{1} \cdot X \text{ and } S'^{1} \cdot X \text{ are equivalent.} \]

The Grundy classes will be split into several classes for $\sim_1$. 

Equivalence classes of $\sim_1$ for the game 0.33

<table>
<thead>
<tr>
<th>Number of paths of length 2 in the subdivided star</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$\cdots$</th>
<th>$2p$</th>
<th>$2p + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1^*$</td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^*$</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0 \leftarrow 1^* \leftarrow 2^*$</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 \leftarrow 2^\square \leftarrow 0 \leftarrow 1^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0 \leftarrow 3^\square \leftarrow 1 \leftarrow 2^\square \leftarrow 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 \leftarrow 2^\square \leftarrow 0 \leftarrow 3^\square \leftarrow 1 \leftarrow 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0 \leftarrow 3^\square \leftarrow 1 \leftarrow 2^\square \leftarrow 0 \leftarrow 3 \leftarrow (03)^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 \leftarrow 2^\square \leftarrow 0 \leftarrow 3^\square \leftarrow 1 \leftarrow 2 \leftarrow (12)^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \cdot^{1} \cdot S_2$ depending on the classes of $S_1$ and $S_2$ is given by:
Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \rightarrow S_2$ depending on the classes of $S_1$ and $S_2$ is given by:

\[
\begin{array}{cccccccc}
0 & 1 & 1^* & 2 & 2^* & 2 \square & 3 & 3 \square \\
\hline
0 & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ \\
1 & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ \\
1^* & ⊕ & ⊕ & 2 & ⊕ & 0 & ⊕ & ⊕ \\
2 & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ \\
2^* & ⊕ & ⊕ & 0 & ⊕ & 1 & 1 & ⊕ & 0 \\
2 \square & ⊕ & ⊕ & ⊕ & ⊕ & 1 & ⊕ & ⊕ & ⊕ \\
3 & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ & ⊕ \\
3 \square & ⊕ & ⊕ & ⊕ & ⊕ & 0 & ⊕ & ⊕ & ⊕ \\
\end{array}
\]

where $⊕$ is the Nim-sum.
Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \cdot 1 \cdot S_2$ depending on the classes of $S_1$ and $S_2$ is given by:

$$
\begin{array}{cccccccc}
& 0 & 1 & 1^* & 2 & 2^* & 2^\square & 3 & 3^\square \\
0 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
1 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
1^* & \oplus & \oplus & 2 & \oplus & 0 & \oplus & \oplus & \oplus \\
2 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
2^* & \oplus & \oplus & 0 & \oplus & 1 & 1 & \oplus & 0 \\
2^\square & \oplus & \oplus & \oplus & \oplus & 1 & \oplus & \oplus & \oplus \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
3^\square & \oplus & \oplus & \oplus & \oplus & 0 & \oplus & \oplus & \oplus \\
\end{array}
$$

where $\oplus$ is the Nim-sum.

$\Rightarrow$ The values are still in the range $[0; 3]$
Equivalence classes of $\sim_2$ for the game 0.33

Number of paths of length 2 in the subdivided star

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$\cdots$</th>
<th>$2p$</th>
<th>$2p+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\Box$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cdots$</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

$0 \Box 3 \Box 1 \Box 2 \Box 0 \Box 3 \Box (03)^* 0$

$1 \Box 2 \Box 0 \Box 3 \Box 1 \Box 2 \Box (12)^* 1 \Box 2$
Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \xtwoheadrightarrow{2} S_2$ depending on the classes of $S_1$ and $S_2$ is given by:

\[
\begin{array}{cccccccccc}
& 0 & 0^* & 1 & 1^* & 1\square & 2 & 2^* & 2\square & 3 & 3\square \\
0 & \oplus & \oplus_1 & \oplus & 2 & \oplus_1 & \oplus & 0 & \oplus_1 & \oplus & \oplus_1 \\
0^* & \oplus_1 & \oplus_1 & \oplus_1 & 2 & \oplus_1 & \oplus_1 & 0 & \oplus_1 & \oplus_1 & \oplus_1 \\
1 & \oplus & \oplus_1 & \oplus & 3 & \oplus_1 & \oplus & 1 & \oplus_1 & \oplus & \oplus_1 \\
1^* & 2 & 2 & 3 & 0 & 3 & 0 & 1 & 1 & 1 & 0 \\
1\square & \oplus_1 & \oplus_1 & \oplus_1 & 3 & \oplus_1 & \oplus_1 & 1 & \oplus_1 & \oplus_1 & \oplus_1 \\
2 & \oplus & \oplus_1 & \oplus & 0 & \oplus_1 & \oplus & 2 & \oplus_1 & \oplus & \oplus_1 \\
2^* & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\
2\square & \oplus_1 & \oplus_1 & \oplus_1 & 1 & \oplus_1 & \oplus_1 & 2 & 0 & \oplus_1 & 1 \\
3 & \oplus & \oplus_1 & \oplus & 1 & \oplus_1 & \oplus & 3 & \oplus_1 & \oplus & \oplus_1 \\
3\square & \oplus_1 & \oplus_1 & \oplus_1 & 0 & \oplus_1 & \oplus_1 & 3 & 1 & \oplus_1 & 0 \\
\end{array}
\]

where $\oplus$ is the Nim-sum and $x \oplus_1 y$ stands for $x \oplus y \oplus 1$. 
Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \bullet \leftrightarrow S_2$ depending on the classes of $S_1$ and $S_2$ is given by:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0*</th>
<th>1</th>
<th>1*</th>
<th>1☐</th>
<th>2</th>
<th>2*</th>
<th>2☐</th>
<th>3</th>
<th>3☐</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>⊕</td>
<td>⊕</td>
<td>⊕</td>
<td>2</td>
<td>⊕</td>
<td>0</td>
<td>⊕</td>
<td>0☐</td>
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<td>⊕</td>
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<tr>
<td>0*</td>
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<td>3</td>
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<tr>
<td>1*</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
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<td>1☐</td>
<td>⊕</td>
<td>0☐</td>
<td>⊕</td>
</tr>
</tbody>
</table>

where ⊕ is the Nim-sum and $x \oplus_1 y$ stands for $x \oplus y \oplus 1$.

⇒ The values are still in the range $[0; 3]$
The game $0.33$ on trees
The game 0.33 on trees

Proposition
The reduction of paths to their length modulo 3 does not work on trees:
Proposition

The reduction of paths to their length modulo 3 does not work on trees:
The game 0.33 on trees

**Proposition**
The reduction of paths to their length modulo 3 does not work on trees:

\[
\begin{array}{c}
\text{\includegraphics[width=2cm]{tree1.png}} \\
\neq \\
\text{\includegraphics[width=2cm]{tree2.png}}
\end{array}
\]

**Conjecture**
For all \( n \geq 4 \), there exists a tree \( T \) such that \( \mathcal{G}(T) = n \).

\[
\mathcal{G}(\text{\includegraphics[width=5cm]{long_tree.png}}) = 10
\]
Conclusion

Summary
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Summary

- Natural generalization of octal games on graphs;

Perspectives

- Prove that trees can have arbitrarily large Grundy values;
- Studying other graph classes;
- Generalize some results on other octal games.
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- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be *reduced* to its *length modulo* 3;
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Merci