Investigating the Mapping between Default Logic and Inconsistency-Tolerant Semantics

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Abstract. In this paper we propose a mapping between inconsistent ontological knowledge bases and semi-monotonic, prerequisite-free closed normal default theories. As a proof of concept of the new results obtained by the proposed mapping we introduce an any-time algorithm for query answering that starts off by a small set of facts and incrementally adds to this set.

1 Introduction

The ONTOLOGY-BASED DATA ACCESS (ODBA) problem [18] investigates querying multiple data sources defined over the same ontology [13]. We distinguish ourselves from other approaches in the OBDA community by considering a rule based language that gains more and more interest from a practical point of view [4]. We consider existential variables in the head of the rules as well as n-ary predicates and conflicts (and generalise certain subsets of Description Logics (e.g. DL-Lite) [1, 5]). The tractability conditions of the considered rule based language rely on different saturation (chase) methods [15]. For algorithmic considerations here we will restrict ourselves to a tractable fragment (such as weakly-acyclic rule sets) [4].

The hypothesis made by OBDA setting is that each data source is assumed to be self-consistent along with the ontology, whereas the integration of homogeneous factual information from all data sources might be no longer consistent [12]. A common solution is to construct a maximal (with respect to set inclusion) consistent subsets of the knowledge base called repair. Once repairs computed, different semantics can be used for query answering over the knowledge base. In this paper we focus on **A**ll **R**epairs semantics (*AR*-semantics) and *Brave*-semantics [2, 3, 12].

Much research has been undertaken in the field of Default Logics and several tools and frameworks have been developed (GaDeL [16], X-Ray [21], DeReS[6]). Moreover, there was an increasing interest in relating Reiter's Default Logic to other non-monotonic formalism such as Minimal Temporal Epistemic Logic [8], Autoepistemic Logics [11, 9], Circumscription [10], Argumentation [7] and the modal logic S4F [22]. Any attempt to find a relation between inconsistency handling in OBDA and Reiter's Default Logic would benefit from such well-established tools and equivalent formalisms. To the best of our knowledge, there is no work on relating both inconsistent ontological knowledge bases semantics and Reiter's Default Logic considering (1) the expressive setting of existential rules [4] and (2) assuming that the inconsistency comes from the set of facts. Our work differs from existing work in the literature considering a similar intuition but only focusing on the propositional case.

The contribution of the paper lies in the following points. We propose an efficient (with respect to time and space complexity) mapping between inconsistent ontological knowledge bases (expressed in a general rule-based language) and a class of default theories (semi-monotonic, precisely *prerequisite-free closed normal default theories*). We formally prove the equivalence between the *inconsistency-tolerant semantics (AR and Brave)* in OBDA and inference in Default Logic *(sceptical and credulous)* in the aforementioned class of default theories. We also formally prove the property of semi-monotonicity for inconsistency-tolerant semantics. This property will serve as a basis for an *anytime algorithm* for query answering.

We show that the non-ontological information (that is information contained in the data sources, also called facts in the remainder of the paper) can be viewed as closed normal defaults. Defaults along with the ontological rules will form a closed normal default theory. We show then the link between default extensions and maximal consistent subsets of facts (called repairs). Based on this link we obtain equivalences between the inconsistency-tolerant semantics *(AR and Brave)* and inference in Default Logic *(sceptical and credulous)*.

From an engineering perspective, our work paves the way for an inconsistency handling hybrid system that incorporates different formalisms at distinct levels. While the algorithm presented in the paper constitutes a proof of concept of the benefit of our mapping, it could be the case that clever application of algorithms for default logic could improve the state of art in OBDA. For example, by bridging the gap between the two fields we can make use of the relation between Reiter's Default Logic and Answer Set Programming [14] for an efficient query answering engine.

2 Preliminaries

Let us briefly recall the basics of the rule-based language used in this paper (equivalent to Datalog$\pm$ [4]), namely, existential rules, negative constraints, facts and knowledge base as well as inconsistency-tolerant semantics.

2.1 Rule-Based Language

We consider *The positive existential* conjunctive fragment of first-order logic, which is composed of formulas built with the connectives $\{\wedge, \rightarrow\}$ and the quantifiers $\{\exists, \forall\}$. We consider first-order vocabularies with constants but no other function symbol. A term t is a constant or a variable, different constants represent different values (unique name assumption), an atomic formula (or atom) is of the form $p(t_1, ..., t_n)$ where p is an n-ary predicate, and $t_1, ..., t_n$ are terms. A *ground* atom is an atom with no variables . Given an atom or a set of atoms A, $vars(A)$, $consts(A)$ and $terms(A)$ denote its set of variables, constants and terms.

An *existential rule* (rule) is a first-order formula of the form $R = \forall \overrightarrow{x} \forall \overrightarrow{y} (H \rightarrow \exists \overrightarrow{w} C)$ where H (resp. C) is a conjunction of atoms called the *hypothesis* (resp. *conclusion*) of R. The variables in the hypothesis (resp. conclusion) are denoted as $vars(H) =$

$\overrightarrow{x} \cup \overrightarrow{y}$ (resp. $vars(C) = \overrightarrow{w}$). Note that, the notation $\overrightarrow{x}$ represents a sequence of variables. We omit quantifiers and we use $R = (H, C)$ as a contracted form of a rule R. An existential rule with an empty hypothesis is called a *fact*. A fact is an existentially closed (with no free variable) conjunction of atoms. i.e. $\exists x(teacher(x) \wedge employee(x))$. This fact allows to assert an unknown individual which is an essential aspect in open-domain perspectives where it cannot be assumed that all individuals are known in advance. A boolean conjunctive query has the same form as a fact. A *negative constraint* is a rule with a conclusion equals to the truth constant false "$\perp$". $N = \forall x, y, z\,(supervises(x, y) \wedge work\_in(x, z) \wedge directs(y, z)) \rightarrow \perp$ means it is impossible for x to supervise y if x works in department z and y directs z.

Given a conjunction of atoms A_1 and A_2, a homomorphism π from A_1 to A_2 is a substitution of $vars(A_1)$ by $terms(A_2)$ such that $\pi(A_1) \subseteq A_2$.

A rule $R = (H, C)$ is *applicable* to a set of atoms A if and only if there exists $A' \subseteq A$ such that there is a homomorphism π from H to the conjunction of elements of A'. For example, the rule $\forall x(teacher(x) \rightarrow employee(x))$ is applicable to the set $\{teacher(Tom), cute(Tom)\}$, since there is a homomorphism from $teacher(x)$ to $teacher(Tom)$. If a rule R is applicable to a set A, the application of R to A according to π produces a set $A \cup \{\pi(C)\}$. In our example, the produced set is $\{teacher(Tom), employee(Tom), cute(Tom)\}$. We then say that the new set (which includes the old one and adds the new information to it) is an *immediate derivation* of A by R. This new set is denoted by $R(A)$. Since facts are conjunction of atoms, given two facts f and f', $f \models f'$ iff there is a homomorphism from f' to f.

Let F be a set of facts and $\mathcal{R}$ be a set of rules. An $\mathcal{R}$-derivation of F is a finite sequence $\langle F_0, ..., F_n\rangle$ s.t $F_0 = F$, and for all $0 \leq i < n$, there is a rule $R_i = (H_i, C_i) \in \mathcal{R}$ and a homomorphism π_i from H_i to F_i s.t $F_{i+1} = F_i \cup \{\pi(C_i)\}$. For a set of facts $\mathcal{F}$ and a query Q and a set of rules $\mathcal{R}$, we say $\mathcal{F}, \mathcal{R} \models Q$ iff there exists an $\mathcal{R}$-derivation $\langle(F_0 = F), ..., F_n\rangle$ such that $F_n \models Q$ [15]. Given a set of facts $\{f_0, \ldots, f_k\}$ and a set of rules $\mathcal{R}$, the closure of $\{f_0, \ldots, f_k\}$ with respect to $\mathcal{R}$, denoted by $\mathtt{Cl}_{\mathcal{R}}(\{f_0, \ldots, f_k\})$, is defined as the smallest set (with respect to $\subseteq$) which contains $\{f_0, \ldots, f_k\}$, and is closed under $\mathcal{R}$-derivation (that is, for every $\mathcal{R}$-derivation $D_i = \langle(F'_1 = \{f_i\}), ..., F'_m\rangle$ of $f_i \in \{f_0, \ldots, f_k\}$ s.t $i \in \{0, ..., k\}$, we have $F'_m \subseteq \mathtt{Cl}_{\mathcal{R}}(\{f_0, \ldots, f_k\})$ and $m \in \mathbb{N}$. Finally, we say that a set of facts $\mathcal{F}$ and a set of rules $\mathcal{R}$ *entail* a fact G (and we write $\mathcal{F}, \mathcal{R} \models G$) iff the closure of the facts by all the rules entails G (i.e. if $\mathtt{Cl}_{\mathcal{R}}(\mathcal{F}) \models G$).

A *knowledge base* $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ is composed of finite set of facts $\mathcal{F}$, finite set of existential rules $\mathcal{R}$ and a finite set of negative constrains $\mathcal{N}$. Given a knowledge base, one can ask if a boolean conjunctive query Q holds or not. The answer to the query Q is *yes* if and only if $\mathcal{F}, \mathcal{R} \models Q$. In this paper we refer to a *boolean conjunctive query* as *query*.

Given a knowledge base $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$, a set $\{f_1, \ldots, f_k\} \subseteq \mathcal{F}$ is said to be *inconsistent* if and only if there exists a constraint $N \in \mathcal{N}$ such that $\{f_1, \ldots, f_k\} \models H_N$, where H_N denotes the hypothesis of the constraint N. A set of facts is consistent if and only if it is not inconsistent. A set $\{f_1, \ldots, f_k\} \subseteq \mathcal{F}$ is $\mathcal{R}$*-inconsistent* if and only if there exists a constraint $N \in \mathcal{N}$ such that $\mathtt{Cl}_{\mathcal{R}}(\{f_1, \ldots, f_k\}) \models H_N$, where H_N is the hypothesis of the constraint N. A set of facts is said to be $\mathcal{R}$*-consistent* if and only if it is not $\mathcal{R}$-inconsistent. A knowledge base $(\mathcal{F}, \mathcal{R}, \mathcal{N})$ is said to be *inconsistent* iff $\mathcal{F}$ is $\mathcal{R}$-inconsistent.

Example 1. Let us consider the following knowledge base $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$, with: $\mathcal{F} = \{teacher(Tom), student(Tom)\}, \mathcal{R} = \{\forall x(teacher(x) \rightarrow university\_member(x)), \forall x(student(x) \rightarrow university\_member(x))\}, \mathcal{N} = \{\forall x(student(x) \wedge teacher(x) \rightarrow \perp)\}$. It is obvious that $\mathcal{K}$ directly violates the constraint $\mathcal{N}$ ($\mathcal{F}$ is $\mathcal{R}$-inconsistent since $\mathtt{Cl}_{\mathcal{R}}(\mathcal{F})$ entails the hypothesis of the negative constraint). Consequently, $\mathcal{K}$ is inconsistent.

2.2 Inconsistency-Tolerant Semantics

Notice that (like in classical logic), if a knowledge base $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ is inconsistent then everything can be entailed from it. A common solution [2, 12] is to construct maximal (with respect to set inclusion) consistent subsets of $\mathcal{K}$. These repairs represent different ways of regaining consistency while maintaining as much information as possible from the original knowledge base. Such subsets are called *repairs*.

Definition 1 (Repair). *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be a knowledge base. A repair $\mathcal{A}$ of $\mathcal{K}$ is an inclusion-maximal subset of $\mathcal{F}$ such that (i) $\mathcal{A}$ is $\mathcal{R}$-consistent, (ii) there exist no $\mathcal{A}'$ such that $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A}'$ is $\mathcal{R}$-consistent. The set of all repairs of $\mathcal{K}$ is denoted by $Repair(\mathcal{K})$.*

Once the repairs are calculated, different semantics can be used for query answering over the knowledge base[1]. For example, we may want to accept a query if it is entailed by *all repairs* (AR-semantics), another possibility is to accept the query if it is entailed by at least some repairs (Brave-semantics). The definitions of the previous semantics are introduced by [2, 12] and adapted for the rule-based language:

Definition 2 (AR-Semantics). *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be a knowledge base and let α be a query. Then α is AR-entailed from $\mathcal{K}$, written $\mathcal{K} \models_{AR} \alpha$ iff for every repair $\mathcal{A}' \in Repair(\mathcal{K})$, it holds that $\mathtt{Cl}_{\mathcal{R}}(\mathcal{A}') \models \alpha$.*

Definition 3 (Brave-Semantics). *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be a knowledge base and let α be a query. Then α is* ***brave-entailed*** *from $\mathcal{K}$, written $\mathcal{K} \models_{Brave} \alpha$ iff $\mathtt{Cl}_{\mathcal{R}}(\mathcal{A}') \models \alpha$ for at least one $\mathcal{A}' \in Repair(\mathcal{K})$.*

Example 2 (Example 1 Cont.). $Repair(\mathcal{K}) = \{\mathcal{A}_1, \mathcal{A}_2\}$ with $\mathcal{A}_1 = \{teacher(Tom)\}$ and $\mathcal{A}_2 = \{student(Tom)\}\}$. $\mathtt{Cl}_{\mathcal{R}}(\mathcal{A}_1) = \{teacher(Tom), university\_member(Tom)\}$, $\mathtt{Cl}_{\mathcal{R}}(\mathcal{A}_2) = \{student(Tom), university\_member(Tom)\}$. For example, we get $\mathcal{K} \models_{\mathtt{AR}} university\_member(Tom)$, $\mathcal{K} \models_{\mathtt{brave}} teacher(Tom)$ and $\mathcal{K} \models_{\mathtt{brave}} student(Tom)$.

[1] These semantics are called inconsistency-tolerant semantics.

2.3 Reiter's Default Logic

We presume a basic familiarity with Default Logic [19] and only recall basic notions. In Default Logic, we represent certain facts about the world in a background theory W, whereas we represent certain rules that express normally and generally in a set D, these rules are called defaults. A default theory $\Delta = (W, D)$ is a pair composed of a background theory W and a set of defaults D expressed in a logical language (we consider first-order logic). A default takes the form of $\delta = A : B/C$, where A, B and C are first-order logic (FOL) formulae (possibly, with free variables) denoted by $pre(\delta), just(\delta)$ and $cons(\delta)$ respectively and standing for **prerequisite**, **justification** and **conclusion** respectively. The default is interpreted as "If it is the case that A and it is consistent to assume B then deduce C". A default δ where $just(\delta) = cons(\delta)$ is called a normal default, if δ has no free variables then it is closed. Moreover, if $pre(\delta) = \emptyset$ we call the default δ prerequisite-free. A default theory is said to be a closed normal default theory if and only if all its defaults are closed and normal. A default theory may induce zero, one, or multiple *extensions*:

Definition 4 (Reiter's extension [19]). *Let $\Delta = (D, W)$ be a default theory. The operator Γ assigns to every set S of formulae the smallest set U of formulae such that: (i) W $\subseteq$ U, (ii) $\mathcal{T}h$(U)=U, (iii) If $(A : B/C) \in D$,U $\vDash$ A, S $\nvDash \neg B$, then C $\in$ U. A set E of formulae is an extension of Δ if and only if $E = \Gamma(E)$, that is, E is a fixed point of Γ.*

Notice that, $\mathcal{T}h$(U) is the deductive closure of U (i.e. the set of logical consequences of a set of formulae U).

Any extension represents a set of acceptable beliefs that can be deduced from an incomplete description of the world described in W. Furthermore, for a given extension E and a set of defaults D, the set of *generating defaults* for E with respect to Δ is $GD(D, E) = \{\delta \in D | E \vDash pre(\delta)$ and $E \nvDash \neg just(\delta)\}$. For a set of defaults $D' \subseteq D$ we denote by Con(D') the set of the conclusions of all defaults in D', Namely, Con(D')={cons(δ) | $\delta \in D'$}. Note that if $\mathcal{G}$ is the set of generating defaults for an extension E then E=$\mathcal{T}h(W \cup Con(\mathcal{G}))$[19].

Reiter defined some inference problems in Default Logic as follows. For a given default theory $\Delta = (W, D)$ a well-formed formula α is *sceptically entailed* from Δ iff α belongs to all extensions of Δ. Whereas it is *credulously entailed* from Δ iff it belongs to at least one extension of Δ.

In this paper we are interested in *closed normal default theories*. This type of default theories is a subset of the so called semi-monotonic default theories that have gained an increasing interest for it desirable properties (it admits local proof procedures as mentioned in [20, 19] and it is implemented in many systems such as X-Ray [17]).

Theorem 1. *[19] Let $\Delta = (W, D)$ be a closed normal default theory, let E and F be two extensions of Δ and let D, D' be set of closed normal defaults s.t $D' \subseteq D$. Then, Δ enjoys the following properties:*

1. *Minimality: if $F \subseteq E$, then E = F.*
2. *Orthogonality: If F and E are distinct then $F \cup E$ is inconsistent.*
3. *Semi-monotonicity: if E' is an extension of $\Delta' = (W, D')$ then $\Delta = (W, D)$ has an extension E s.t $E' \subseteq E$ and $GD(D', E') \subseteq GD(D, E)$.*

Theorem 2. *[19] Let $\Delta = (W, D)$ be a closed normal default theory such that $D' \subseteq D$. Suppose that E'_1 and E'_2 are distinct extensions of (W, D'). Then Δ has distinct extensions E_1 and E_2 such that $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$.*

The property of *semi-monotonicity* in Theorem 1 stipulates that a closed normal default theory is *monotone* with respect to the addition of new defaults. Theorem 2 implies that the addition of new closed normal defaults to a closed normal default theory can never lead to a default theory with fewer extensions than the original.

3 A Default Logic Interpretation of Inconsistent Knowledge Base

In the OBDA setting we consider the set of facts as the source of inconsistency. Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be a knowledge base, the intuition underlying the mapping is that we handle all the facts according to the principle "every fact in $\mathcal{K}$ is consistent with the rest unless proven inconsistent". That is, we transform each fact in $\mathcal{F}$ into a default while maintaining the set $\mathcal{R}$ and $\mathcal{N}$ since they are assumed to be consistent. Specifically, if we have a fact α in our knowledge base $\mathcal{K}$ we consider it as "generally consistent" until we prove the other way around by developing our initial knowledge in W using defaults. This leads to the following:

Definition 5 (Mapping τ). *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be an inconsistent knowledge base and $\Delta_{\mathcal{K}} = (W, D)$ be a closed normal default theory. Furthermore, Let $\mathcal{S}$ be the set of all possible inconsistent knowledge bases and $\mathcal{D}$ be the set of all possible closed normal default theories. The mapping τ is defined from $\mathcal{S}$ to $\mathcal{D}$ such that $\tau(\mathcal{K}) = \Delta_{\mathcal{K}}$ as follows:*

(a) The mapping τ associates for every fact $f_i \in \mathcal{F}$ its default $\delta_{f_i} =: f_i/f_i$ in D. Notice that δ_{f_i} is a prerequisite-free, closed normal default since facts in $\mathcal{K}$ are either ground atoms or existentially closed conjunction of atoms (see Section 2.1).

(b) The mapping τ associates for every rule $R_i \in \mathcal{R}$ ($N_i \in \mathcal{N}$, resp.) the same $\mathcal{R}_i$ (N_i, resp.) in W.

One major implication of the mapping is that every inconsistent knowledge base can be efficiently (w.r.t time and space) mapped to a closed normal default theory. The benefit of such mapping is that we can handle inconsistency issues with a theoretically and practically well-established framework (i.e. Default Logic). In what follows, we present the application of mapping on Example 1.

Example 3. Consider $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ of example 1, the corresponding default theory $\tau(\mathcal{K}) = \Delta_{\mathcal{K}} = (D, W)$ is a closed normal default theory with a background knowledge:

$W = \{\forall x(teacher(x) \rightarrow university\_member(x)), \forall x(student(x) \rightarrow university\_member(x)), \forall x(teacher(x) \wedge student(x) \rightarrow \bot)\}$.

And a set of defaults:

- $D = \{: teacher(Tom)/teacher(Tom), : student(Tom)/student(Tom)\}$.

With a set of extensions $Ext(\Delta_{\mathcal{K}}) = \{E_1, E_2\}$ such that:

- $E_1 = \mathcal{T}h(W \cup \{teacher(Tom)\}) = \mathcal{T}h(\{teacher(Tom), university\_member(Tom)\})$.
- $E_2 = \mathcal{T}h(W \cup \{student(Tom)\}) = \mathcal{T}h(\{student(Tom), university\_member(Tom)\})$.

The set of generating defaults for E_1 (resp. E_2) is the singleton $GD(D, E_1) = \{: teacher(Tom)/teacher(Tom)\}$ (resp. $GD(D, E_2) = \{: student(Tom)/student(Tom)\}$).

4 Equivalences and New Results

As stated above, the previous mapping establishes a relation between inconsistent knowledge bases and closed normal default theories. We show how this relation also holds for repairs and extensions; consequently between inconsistency-tolerant semantics and inference in Default Logic.

Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be an inconsistent knowledge base and $\Delta_{\mathcal{K}} = (D, W)$ the corresponding default theory using the mapping τ. Let $\texttt{Ext}(\Delta_{\mathcal{K}})$ be the set of all extensions of $\Delta_{\mathcal{K}}$ and $\mathcal{R}epair(\mathcal{K})$ the set of all repairs of $\mathcal{K}$.

Proposition 1. *For every extension $E_i \in \texttt{Ext}(\Delta_{\mathcal{K}})$ it holds that the set $Con(GD(D, E_i)) \in \mathcal{R}epair(\mathcal{K})$.*

Proof 1 *First, notice that the set Con(D)=$\mathcal{F}$, since every fact in $\mathcal{F}$ has been mapped to a default.*

Let us first prove that the for every $E_i \in \texttt{Ext}(\Delta_{\mathcal{K}})$ the set Con(GD(D,E_i)) is consistent (contradiction free). On the one hand, we have Con(GD(D,E_i)) $\subset E_i$. On the other hand, since E_i is an extension then it has to be consistent (E $\nvDash \bot$). Hence, Con(GD(D,E_i)) has to be consistent. By means of contradiction we prove now that for every E_i $\in$Con(GD(D,E_i)) is maximally (w.r.t $\subseteq$) consistent subset of $\mathcal{F}$. Let us suppose the contrary, i.e. Con(GD(D,E)) is not maximally consistent. Then there exists another extension E' for which Con(GD(D,E_i)) $\subseteq$ Con(GD(D,E')) such that GD(D,E') is the set of generating defaults of E'. Consequently, $\mathcal{T}h$(W $\cup$ Con(GD(D,E_i))) $\subseteq$ Cn(W $\cup$ Con(GD(D,E'))). In other words, $E_i \subseteq E'$ is in contradiction with Theorem 1 (minimality).

Example 4 (Example 3 cont.). Consider $\Delta_{\mathcal{K}}$, we have $Con(GD(D, E_1)) = \{teacher(Tom)\}$ and $Con(GD(D, E_2)) = \{student(Tom)\}$, it is obvious that $Con(GD(D, E_1)) \in \mathcal{R}epair(\mathcal{K})$ and $Con(GD(D, E_2)) \in \mathcal{R}epair(\mathcal{K})$.

On one hand, extensions in $\Delta_{\mathcal{K}}$ represent a maximal set of beliefs that can be together. On the other hand, repairs also represent a maximal non-conflicting set of facts. Thus there is a relation between extensions and repairs.

Proposition 2. *Every extension E contains one and only one repair $\mathcal{A} \in \mathcal{R}epair(\mathcal{K})$.*

Proposition 3. *Every repair $\mathcal{A} \in \mathcal{R}epair(\mathcal{K})$ is contained in one and only one extension E $\in Ext(\Delta_{\mathcal{K}})$.*

Proof 2 *First, we prove that every repair is contained in at least one extension. Next, we prove that every repair is contained in only one extension.*

1. *On one hand we have $\mathcal{A} \subseteq \mathcal{F}$ thus $R \subseteq Con(D)$ (there exists a set of defaults $D' \subseteq D$ such that $\mathcal{A} = Con(D')$). On other hand, since $\mathcal{A}$ is maximally consistent (w.r.t $\subseteq$), by definition we get $\forall f_i \in \mathcal{F}-\mathcal{A}$, $\mathtt{Cl}_{\mathcal{R}}(\mathcal{A} \cup f_i)$ is $\mathcal{R}$-inconsistent, similarly, $\forall d_i \in D - D'$ the set $\mathcal{T}h(W \cup \mathcal{A} \cup cons(d_i)) \vDash \perp$. Therefore we conclude that; (1) $E = \mathcal{T}h(W \cup \mathcal{A})$ is closed under D' (there is no default that can be applied), (2) $W \subseteq E$; (3) $\mathcal{T}h(E) = E$. Consequently, E is an extension (according to Definition 4) such that $\mathcal{A} \in E$.*
2. *Now let us prove that E is the only extension that contains $\mathcal{A}$. Suppose that E and E' are two extensions and $\mathcal{A} \in E$ and $\mathcal{A} \in E'$, from (1) we have $E = \mathcal{T}h(W \cup \mathcal{A})$ and $E' = \mathcal{T}h(W \cup \mathcal{A})$, it is not hard to see that E and E' are not orthogonal which contradicts Theorem 1.*

From 1 and 2 we conclude that a repair $\mathcal{A}$ is contained in one and only one extension.

In the following example we show the relation between extensions and repairs.

Example 5 (Example 3 count.). Consider the repairs $\mathcal{A}_1$ and $\mathcal{A}_2$ of example 2 and the extensions E_1 and E_2 of the corresponding default theory $\Delta_{\mathcal{K}}$. One can clearly see that $E_1 = \mathcal{T}h(\{teacher(Tom), university\_member(Tom)\}) \supset \mathcal{A}_1 = \{teacher(Tom)\}$ and $E_2 = \mathcal{T}h(\{student(Tom), university\_member(Tom)\}) \supset \mathcal{A}_1 = \{student(Tom)\}$.

Previous propositions in this section show that the mapping τ induces a link between *repairs* and *extensions*. We use this link to prove the relation between *AR-semantics* and *sceptical* inference and also between the *Brave-semantics* and *credulous inference* in closed normal default theories.

Theorem 3. *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be an inconsistent knowledge base, $\Delta_{\mathcal{K}} = (W, D)$ be the corresponding closed normal default theory obtained by the mapping τ and let α be a query. Then:*

1. *$\mathcal{K} \vDash_{\mathtt{AR}} \alpha$ iff α is sceptically entailed by $\Delta_{\mathcal{K}}$.*
2. *$\mathcal{K} \vDash_{\mathtt{brave}} \alpha$ iff α is credulously entailed by $\Delta_{\mathcal{K}}$.*

In Example 2, we mentioned that the query $\alpha = university\_member(Tom)$ is *AR*-entailed by the knowledge base $\mathcal{K}$ ($\mathcal{K} \vDash_{\mathtt{AR}} \alpha$). In the next example we show that α is sceptically entailed by Δ_K.

Example 6 (Example 3 count.). Consider the extensions E_1 and E_2 and let $\alpha_1 = university\_member(tom)$ and $\alpha_2 = teacher(tom)$. It is clear that $\alpha_1 \in E_1$ and $\alpha_1 \in E_2$, thus α is sceptically entailed by $\Delta_{\mathcal{K}}$. Whereas α_1 belongs only to E_1 thus α_1 is credulously entailed by $\Delta_{\mathcal{K}}$.

Note that a closed normal default theory always has an extension, and if it has more than one extension then the extensions are inconsistent together (orthogonal). Based on this we can observe that if the corresponding default theory $\Delta_{\mathcal{K}}$ has more than one extension then $\mathcal{K}$ is inconsistent.

By virtue of the equivalences provided in this section, we can prove interesting properties about inconsistency-tolerant semantics. In Section 2 we have shown that closed normal default theories enjoy the property of semi-monotonicity which states that the extensions of the original theory are always preserved within the extension of the new theory (with the addition of new closed normal defaults). We show how this property holds in inconsistency-tolerant semantics.

Theorem 4 (Semi-monotonicity). *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ and $\mathcal{K}' = (\mathcal{F}', \mathcal{R}, \mathcal{N})$ be two inconsistent knowledge bases such that $\mathcal{F}' \subseteq \mathcal{F}$ and let $\mathcal{A}'$ be a repair of $\mathcal{K}'$. Then, $\mathcal{K}$ has a repair $\mathcal{A} \in \mathcal{R}epair(\mathcal{K})$ such that $\mathcal{A}' \subseteq \mathcal{A}$.*

Proof. Let us suppose the following, $\mathcal{K}' = (\mathcal{F}', \mathcal{R}, \mathcal{N})$ and $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ such that $\mathcal{F} = \mathcal{F}' \cup \{f\}$ (we get $\mathcal{K}$ by adding a new fact to $\mathcal{F}'$), suppose further that $\mathcal{A}_1 \in \mathcal{R}epair(\mathcal{K})$ and $\mathcal{A}'_1 \in \mathcal{R}epair(\mathcal{K}')$, then either (1) $\mathcal{A}_1 = \mathcal{A}'_1 \cup \{f\}$ is consistent, then $\mathcal{A}_1$ is a repair of $\mathcal{K}$ such that $\mathcal{A}'_1 \subset \mathcal{A}_1$; or (2) $\mathcal{A}_1 = \mathcal{A}'_1 \cup \{f\}$ is inconsistent, thus $\{f\}$ will form a new repair A_2 such that $\{f\} \in A_2$. In two cases there exists always a repair $\mathcal{A}_1$ such that $\mathcal{A}'_1 \subseteq \mathcal{A}$.

The next corollary then follows.

Corollary 1. *Let $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ and $\mathcal{K}' = (\mathcal{F}', \mathcal{R}, \mathcal{N})$ be two inconsistent knowledge bases such that $\mathcal{F}' \subseteq \mathcal{F}$ and let $\mathcal{A}'_1$ and $\mathcal{A}'_2$ be two repairs of $\mathcal{K}'$. Then $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ has two repairs $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}'_1 \subseteq \mathcal{A}_1$ and $\mathcal{A}'_2 \subseteq \mathcal{A}_1$.*

This stipulates that for an inconsistent knowledge base $\mathcal{K}$, $\mathcal{K}$ is monotone with respect the addition of new facts. That means the repairs of a knowledge base with fewer facts from the original knowledge base are always preserved within the repairs of the original knowledge base. Notice that, $\mathcal{K}$ is non-monotone with respect to the addition of new constraints, because the added constraints can alter its repairs.

In what follows we show the result of Corollary 1 and Theorem 4 on AR and Brave semantics.

Proposition 4. *Let $\mathcal{K}' = (\mathcal{F}', \mathcal{R}, \mathcal{N})$ and $\mathcal{K} = (\mathcal{F}, \mathcal{R}, \mathcal{N})$ be two inconsistent knowledge bases such that $\mathcal{F}' \subseteq \mathcal{F}$ and let α be a query. Then, (i) if $\mathcal{K}' \vDash_{\texttt{brave}} \alpha$ then $\mathcal{K} \vDash_{\texttt{brave}} \alpha$; (ii) if $\mathcal{K}' \vDash_{\texttt{AR}} \alpha$ then $\mathcal{K} \vDash_{\texttt{brave}} \alpha$.*

5 Conclusion

We have studied the relation between Default Logic and inconsistent ontological knowledge bases within a rule-based language in the OBDA setting and shown that every inconsistent knowledge base can be efficiently represented as a closed normal default theory. This gives the possibility to bridge two different formalisms. We proved that inconsistency-tolerant semantics enjoys the same property of a closed normal default theory, namely semi-monotonicity. In addition, this work shows the expressiveness of Default Logic as a powerful non-monotonic formalism that is capable of handling different problems. A further study on the equivalences with another variants of Default Logic (constrained, justified, rational, etc) will be a matter of interest; as well as the relation with Answer Set Programming given its computational efficiency.

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