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On digital plane preimage structure

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Abstract

In digital geometry, digital straightness is an important concept both for practical motivations and theoretical interests. Concerning the digital straightness in dimension 2, many digital straight line characterizations exist and the digital straight segment preimage is well known. In this article, we investigate the preimage associated to digital planes. More precisely, we present first structure theorems that describe the preimage of a digital plane. Furthermore, we present a bound on the number of preimage faces under some given hypotheses.

Keywords: Digital plane preimage; Digital straight line; Dual transformation

1. Introduction

Digital straightness is an important concept in computer vision. In dimension two, for nearly half a century many digital straight line characterizations have been proposed with interactions with many fields such as arithmetic or theory of words (refer to [14] for a survey on digital straight line). A classical way to define a digital straight line is to consider the digitization of an Euclidean straight line on a unit grid. Hence, given a finite subset of a digital straight line, called digital segment, we can characterize the set of Euclidean straight lines whose digitization contains the digital straight segment. Many authors have discussed

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about this set of straight lines, also called *preimage*, of a digital segment [4,9,10]. An important result is that such a preimage is a convex polygon in the parameter space and this domain has got an important arithmetical structure that limits to four the number of vertices. This result of interest for two reasons: on the one hand we have a better understanding of this simple digital object and on the other hand, we can design very efficient digital straight line recognition algorithms. Concerning digital planes, some algorithms exist in order to decide if a set of grid points in dimension three is a part of a digital plane [5,7,8,15,16]. However, no result has been proposed concerning the structure of the digital plane preimage. In this article, we present several results that describe the faces and the vertices of the preimage polyhedron in the parameter space.

In Section 2, we present major results on the digital straight line preimages. The structure theorems for straight lines are then used to characterize digital plane preimage in Section 3. Finally, we present in Section 4 a bound on the number of faces of the digital plane preimage under some given hypotheses.

2. Digital straight line preimage

In the following, we use the notations proposed by Lindenbaum and Bruckstein [9]. Consider a straight line $y = \alpha_0 x + \beta_0$ (without loss of generality, $(\alpha_0, \beta_0) \in [0, 1] \times [0, 1[$). The digitization of this line using the Object Boundary Quantization (see [6] for a survey on digitization schemes) on an unit grid is the set of discrete points such that $L_0 = \{(x, y) \in \mathbb{Z}^2 \mid \lfloor \alpha_0 x + \beta_0 - y \rfloor = 0\}$.

The preimage of a set of pixels S is defined by the set of straight lines whose digitization contains S. The preimage of S, denoted $D(S)$, is the set of (α, β) in the straight line parameter space satisfying:

$$D(S) = \{(\alpha, \beta) \in [0, 1] \times [0, 1[\mid \forall (x, y) \in S,\ y \leqslant \alpha x + \beta < y + 1\}. \tag{1}$$

If we decompose the previous equation, each pixel introduces two linear inequalities. Hence, the preimage of a set of pixels is given by intersection of the linear inequalities associated to each pixel in the parameter space. If such an intersection is non-empty, S is a digital straight segment. Indeed, in that case there exist (α_0, β_0) such that $S \subset L_0$.

Many works have been done concerning the preimage analysis. In the following, we recall properties presented by Dorst and Smeulders [4], McIlroy [10] and Lindenbaum and Bruckstein [9].

Proposition 1. *Let S be a 8-connected set of pixels, the domain $D(S)$ is either empty or a convex polygon in the parameter space with at most four vertices. If $D(S)$ has four vertices, two of them have the same α coordinate which is between the α coordinates of the other two vertices.*

Fig. 1 illustrates all the possible shapes of $D(S)$ (see [9] and a simple proof can be found in [10]).

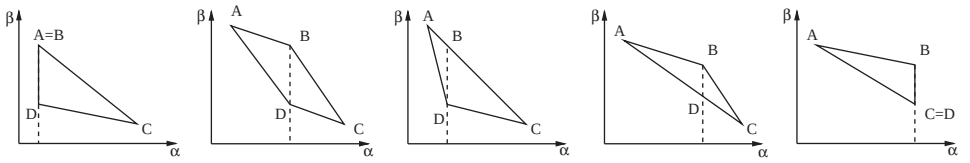

Fig. 1. Five possible shapes of the preimage $D(S)$ of a digital straight segment.

Among all the various definitions of DSS, we retain the one proposed by Reveillès [12] and based on the following definition:

Definition 2. An arithmetical naive straight line, denoted $N(a, b, \mu)$, with $a, b, \mu \in \mathbb{Z}$ and $\gcd(a, b) = 1$ is defined by the set of pixels satisfying:

$$N(a, b, \mu) = \{(x, y) \in \mathbb{Z}^2 \mid \mu \leqslant ax - by < \mu + \max(|a|, |b|)\}. \tag{2}$$

a/b is the slope of the digital line and μ is the lower bound.

If we consider a naive straight line such that $0 \leqslant a < b$ and thus $\max(|a|, |b|) = b$, we have an equivalence between this characterization and the previous one:

Theorem 3 (*Reveillès [12]*). *For all α_0 and β_0 in $[0, 1] \times [0, 1[$, and all finite set $S \subset L_0$, there exist $a, b, \mu \in \mathbb{Z}$ with $0 \leqslant a < b$ such that $S \subset N(a, b, \mu)$.*

In fact, many parameters $\{(a_i, b_i, \mu_i)\}$ such that $S \subset N(a_i, b_i, \mu_i)$ exist. However, we can only consider the minimal set of parameters (a, b, μ) such that $b = \min(\{b_i\})$. Note that the output of the arithmetical naive segment recognition algorithm proposed in [3] is the minimal set of parameters. In the following, when we consider a subset S of a naive straight line $N(a, b, \mu)$, we suppose that (a, b, μ) is the minimal set of parameters for S.

We choose the Reveillès digital straight line representation scheme because it allows simple illustration of the geometry in the primal space of preimage vertices. More precisely, we can define some characteristic points, called *leaning points*, defined as follows: *upper leaning points* (resp. *lower leaning points*) of a digital straight line $N(a, b, \mu)$ are grid points (x, y) satisfying $ax - by = \mu$ (resp. $ax - by = \mu + \max(|a|, |b|) - 1$).

Given a finite connected arithmetical naive segment S with minimal parameters $N(a, b, \mu)$, the segment contains at least three leaning points: one lower leaning point, one upper leaning point and any third one (see [3]). Let us first suppose that S has the following leaning points: U (resp. U′) the upper leaning point of S with minimum x coordinate (resp. maximum x coordinate), and L and L' that are defined in the same way from lower leaning points. Fig. 2(a) illustrates these definitions. Using these arithmetical digital lines, the preimage vertices can be expressed using U, U', L, L' (see Fig. 2):

- the vertex D corresponds to the straight line (UU') in the primal space;
- the vertex B corresponds to the straight line $(L_{\mathrm{T}}L'_{\mathrm{T}})$ where L_{T} (resp. L'_{T}) is L (resp. L') translated by the vector $(0, 1)^{\mathrm{T}}$;

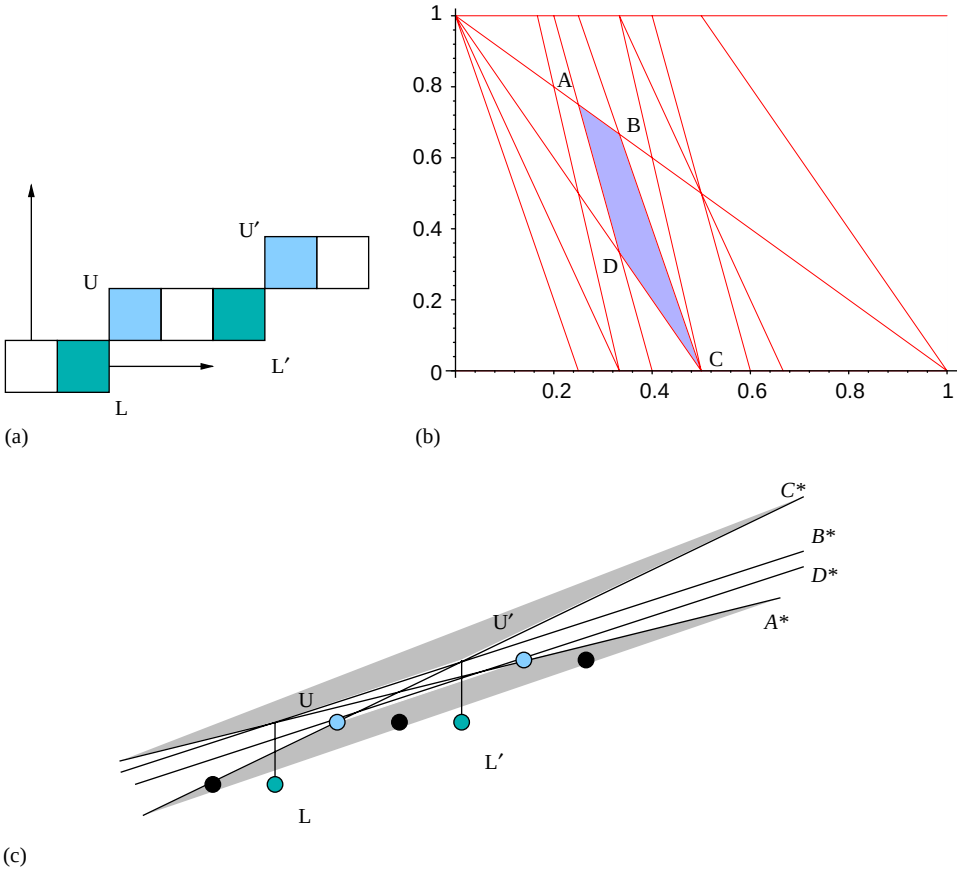

Fig. 2. Illustration in the primal space of the preimage vertices using the arithmetical digital line formalism: (a) a piece of the arithmetical digital straight line $N(1, 3, 1)$ with lower and upper leaning points, (b) its associated preimage and (c) illustration in the primal space of the preimage vertices.

- the vertex A corresponds to the straight line $(L_{\mathrm{T}}U')$;
- the vertex C corresponds to the straight line $(L'_{\mathrm{T}}U)$;
- the coordinates of D and B are, respectively, $(a/b, \mu/b)$ and $(a/b, (\mu+1)/b)$.

If one of the four leaning points is missing, similar results can be derived and the preimage $D(S)$ has only three vertices. In a digital line recognition point of view, the following results can be stated from [12,9]:

Lemma 4. *Let S be a connected arithmetical naive segment with a minimal set of parameters (a, b, μ). Consider a pixel p connected at the left (or right) side of S and such that p belongs to this straight line. The preimage of $S \cup \{p\}$ grid remains unchanged if and only if p is not a leaning point of $N(a, b, \mu)$.*

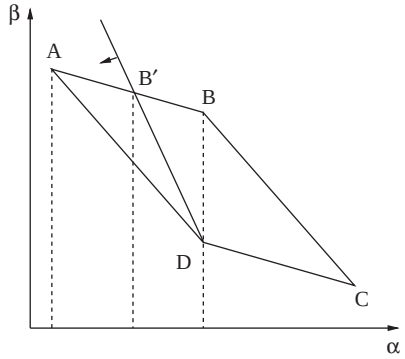

Fig. 3. Any cutting line goes through point B or D [9].

Proof. Let us denote $D(S)$ the preimage of the digital naive line S. We refer to Fig. 1 for the description of $D(S)$. In [9], the authors prove that only one of the two constraints associated to p can modify $D(S)$. Furthermore, they also prove that if the cutting line crosses $D(S)$, it goes through the vertices B or D (see Fig. 3). Since the segment contains at least three leaning points, we have $B = (a/b, \mu/b)$ and $D = (a/b, (\mu+1)/b)$. According to the links between digital naive lines and the preimage structure detailed above, only leaning points of $N(a, b, \mu)$ can have a constraint that goes through B or D. □

3. Digital plane preimage

3.1. Notations and definitions

Without loss of generality, we consider the Euclidean plane given by the parameters $(\alpha_0, \beta_0, \gamma_0) \in [0, 1]^2 \times [0, 1[$. The digitization P_0 of this Euclidean plane is the set of grid points (called *voxels* in 3D) satisfying:

$$P_0 = \{(x, y, z) \in \mathbb{Z}^3 \mid \lfloor \alpha_0 x + \beta_0 y + \gamma_0 - z \rfloor = 0\}. \tag{3}$$

In the same manner as in 2D, let S be a set of voxels. We can define the preimage of S considering the set of parameters (α, β, γ) such that the digitization of the associated plane contains S:

$$D_{3D}(S) = \{(\alpha, \beta, \gamma) \in [0, 1]^2 \times [0, 1[\mid \forall (x, y, z) \in S,\ z \leqslant \alpha x + \beta y + \gamma < z + 1\}. \tag{4}$$

The preimage, denoted $D_{3D}(S)$, is either empty or a convex polyhedron in the (α, β, γ)-parameter space. Indeed, it is the intersection of linear inequalities. We also consider a characterization of the digital plane based on the *arithmetical naive plane* [12]:

Definition 5. An arithmetical naive plane, denoted $P(a, b, c, \mu)$, with $a, b, c, \mu \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$ is defined as the set of voxels satisfying

$$P(a, b, c, \mu) = \{(x, y, z) \in \mathbb{Z}^3 \mid \mu \leqslant ax + by + cz < \mu + \max(|a|, |b|, |c|)\}, \qquad (5)$$

where $(a, b, c)^{\mathrm{T}}$ is the digital plane normal vector and μ is the lower bound.

In the following, we consider a naive plane such that $0 \leqslant a \leqslant b < c$ and thus $\max(|a|, |b|, |c|) = c$. As in dimension 2, for each finite subset of digital plane $S \subset P_0$ given by $(\alpha_0, \beta_0, \gamma_0)$, there exist $a, b, c, \mu \in \mathbb{Z}$ with $0 \leqslant a \leqslant b < c$ such that $S \subset P(a, b, c, \mu)$. Among all possible naive plane parameterizations such that $S \subset P(a, b, c, \mu)$, we only consider the minimal set of parameters such that c is minimal.

In this arithmetical plane, we can also define special voxels, so-called *upper and lower leaning points*: the upper leaning points are the voxels satisfying $ax + by + cz = \mu$ and lower leaning points are the voxels satisfying $ax + by + cz = \mu + \max(|a|, |b|, |c|) - 1$. Since these points are coplanar, we also define the *upper leaning polygon*, denoted L_{up} (resp. *lower leaning polygon* denoted L_{low}) as the 2D convex hull of upper leaning points (resp. lower leaning points). Fig. 4 illustrates these definitions.

Given a finite set piece of digital plane S, if (a, b, c, μ) is the minimal set of parameters for S, then S contains, at least, four leaning points of $P(a, b, c, \mu)$ (two upper and two lower points, three upper and one lower points or one upper and three lower points) [2]. In the following, when a digital naive plane parametrization is considered for a set of voxels, we always suppose that the parametrization is minimal.

In the next section, we present links between preimage faces and leaning polygon vertices.

3.2. Digital plane preimage characterization

First of all, we suppose that the finite subset of digital plane $S \subset P(a, b, c, \mu)$ contains at least three upper leaning points and three lower leaning points (this point will be discussed below). Thus, we first introduce vertices and faces of $D_{3\mathrm{D}}(S)$ given by leaning polygons:

Proposition 6. *Let $S \subset P(a, b, c, \mu)$ be a piece of naive plane. Then, the polyhedron containing all the Euclidean planes $D_{3\mathrm{D}}(S)$ in the parameter space has the following properties*:

- *two particular vertices with coordinates $L^*_{\mathrm{low}}(a/c, b/c, \mu/c)$ and $L^*_{\mathrm{up}}(a/c, b/c, (\mu + 1)/c)$ are identified. They correspond to the planes containing the leaning polygons L_{up} and L_{low} in the primal space*;
- *the polyhedron's faces adjacent to L^*_{low} (resp. L^*_{up}) result from the lower (resp. upper) leaning polygon's vertices.*

Proof. In the (α, β, γ)-parameter space, each point $p(x, y, z)$ in S introduces two linear constraints $C_1(p) : \alpha x + \beta y + \gamma - z \geqslant 0$ and $C_2(p) : \alpha x + \beta y + \gamma - z - 1 < 0$ with $(\alpha, \beta, \gamma) \in [0, 1]^2 \times [0, 1[$. Since (α, β, γ) are positive and according to Preparata and Shamos [11], the domain $D_{3\mathrm{D}}(S)$ is given by computing the lower envelope of constraints

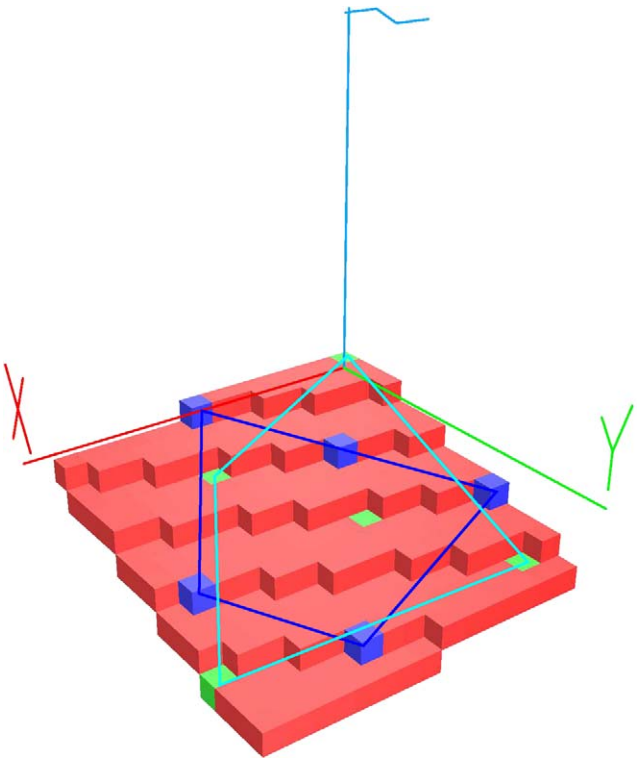

Fig. 4. Illustration of an arithmetical digital plane $P(7, 17, 57, 0)$, lower and upper leaning points and lower and upper leaning polygons with $n = 15$.

C_2, the upper envelope of constraints C_1, and by merging these two envelopes. In other words, we can independently analyze constraints C_1 and C_2.

Hence, we consider the constraints $\{C_2\}$ of points p_i in S and the leaning plane containing the upper leaning polygon L_{up}. Since all points p_i are below the leaning plane by definition of this plane, all half-planes defined by the constraints $\{C_2(p_i)\}$ contain the point L^*_{up} in the parameter space. Thus, since all upper leaning points have constraints C_2 going through L^*_{up}, L^*_{up} is necessarily a vertex of the lower envelope of constraints $\{C_2\}$ and so, L^*_{up} is necessarily a vertex of the polyhedron $D_{3\text{D}}(S)$. Using same arguments, we prove that L^*_{low} is a vertex, in the parameter space, of the upper envelope of constraints $\{C_1(p_i)\}$ and thus, L^*_{low} is also a vertex of $D_{3\text{D}}(S)$. The coordinates of L^*_{up} and L^*_{low} are given by definition of leaning points.

If we consider now the adjacent faces to the point L^*_{up} of $D_{3\text{D}}(S)$, each face with normal vector $(x_i, y_i, z_i)^{\text{T}}$ is created by the upper leaning point with coordinates (x_i, y_i, z_i). We denote $\{e^i\}_{1\ldots m}$ the vertices of the leaning polygon L_{up} and by v a coplanar voxel to points $\{e^i\}_{1\ldots m}$, inside the polygon. Since L_{up} is the planar convex hull of upper leaning points, we have $v = \sum_{i=1}^{m} \omega_i e^i$, where $\{\omega_i\}_{1\ldots m} \in \mathbb{R}$ are such that $\omega_i \geqslant 0$ and $\sum_{i=1}^{m} \omega_i = 1$. Then,

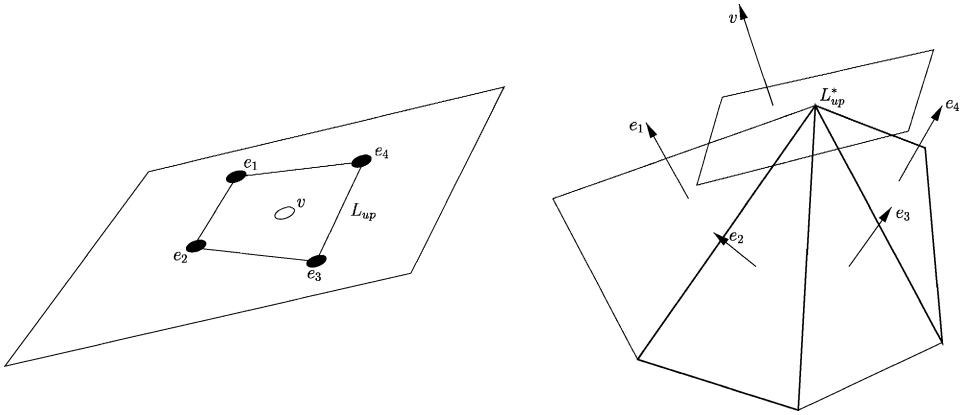

Fig. 5. Illustration of Proposition 6: (Left) vertices $\{e^i\}_{1\ldots m}$ of the upper leaning polygon and the point v lying inside this polygon, (Right) the constraint generated by v has a normal vector linearly dependent with positive weights on normal vectors of faces $\{e^i\}_{1\ldots m}$ in the parameter space.

the constraint generated by v in the dual space contains L^*_{up} and has a normal vector which is linearly dependent with positive weights to normal vectors of faces $\{e^i\}_{1\ldots m}$ (see Fig. 5). Thus, v is not an adjacent face to L^*_{up} in $D_{3\text{D}}(S)$. Finally, all the adjacent faces to L^*_{up} are only generated by the upper leaning polygon's vertices. Similarly, all the adjacent faces to L^*_{low} in the parameter space are generated by lower leaning polygon's vertices. □

At this point, we have proved that the preimage $D_{3\text{D}}(S)$ has two characteristic vertices associated to the leaning planes and particular faces created by the leaning polygon's vertices. We can now discuss on pathological cases when the number of upper (resp. lower) leaning points is less than 3. If only two upper (resp. lower) leaning points exist, the two associated faces belong to $D_{3\text{D}}(S)$ and define an edge of $D_{3\text{D}}(S)$ that can be viewed as a degeneracy of L^*_{up} (resp. L^*_{low}). If there is only one upper leaning point, only the associated face belongs to $D_{3\text{D}}(S)$. In the following, we prove that, with some hypotheses on the digital plane, the preimage $D_{3\text{D}}(S)$ does not contain other faces.

Definition 7. Let $S \subset P(a, b, c, \mu)$ be a piece of naive plane. We define the *double cone* in the parameter space associated to S and denote by $D_{\text{cone}}(S)$ the domain where faces are generated by leaning polygons' vertices.

For the sake of clarity, we suppose in the following that the leaning polygons have at least three points.

As a corollary of Proposition 6 and whatever the digital plane, we have

$$D_{3\text{D}}(S) \subseteq D_{\text{cone}}(S). \tag{6}$$

The following theorems show that we have an equality between these two polyhedra if we add some hypotheses on S.

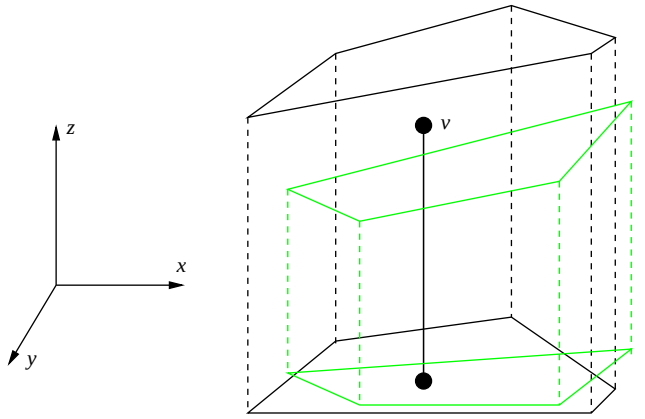

Fig. 6. Illustration of Theorem 8 hypotheses: the projection of the voxel v must belong to the projections of the two leaning polygons.

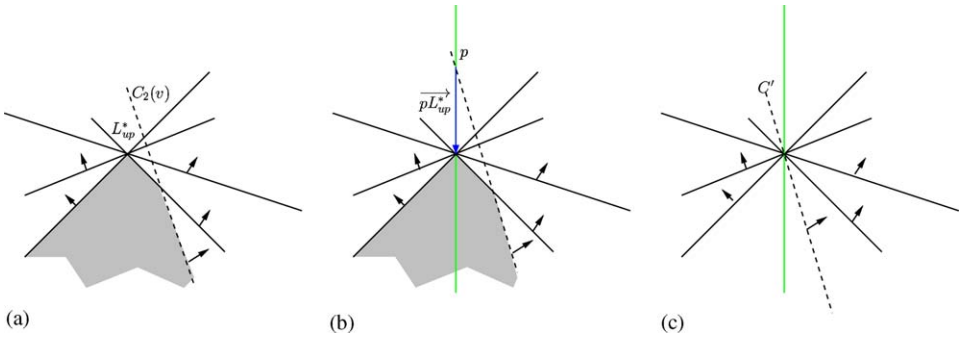

Fig. 7. Illustration of the proof of Lemma 8 in the 2D case.

Theorem 8. *Let $S \subset P(a,b,c,\mu)$ (with $0 \leqslant a \leqslant b < c$) be a piece of naive plane where each point (x_i, y_i, z_i) is such that (x_i, y_i) lies inside the projections onto the plane $z = 0$ of the two leaning polygons. Then we have $D_{3D}(S) = D_{cone}(S)$.*

Proof. Let us consider a voxel v that belongs to the digital plane $P(a,b,c,\mu)$ and that satisfies theorem hypothesis. We first consider the constraint $C_2(v)$ and show that $C_2(v)$ does not intersect $D_{cone}(S)$. The Fig. 6 illustrates the hypotheses of the theorem.

Since v belongs to S, $C_2(v)$ necessarily contains the point $L^*_{up}(a/c, b/c, (\mu+1)/c)$ in the parameter space. In other words, the plane $C_2(v)$ crosses the straight lines $(L^*_{low}L^*_{up})$ at a point p with γ-coordinate greater than the γ-coordinate of L^*_{up} (see Fig. 7). If $C_2(v)$ crosses the domain $D_{cone}(S)$, then the translation of $C_2(v)$ by the vector $\overrightarrow{pL^*_{up}}$ crosses the domain too. This transformation translates the plane $C_2(v)$ into a plane C' that goes through L^*_{up}. In the primal space, this translation corresponds to a vertical projection of the voxel v onto the

upper leaning plane. According to the hypothesis on the digital plane voxels, this vertical projection of v lies inside the upper leaning polygon. Thus, using the same arguments as in the proof of Proposition 6, the normal vector of C' is linearly dependent with positive weights of the face normal vectors created by the upper leaning polygon's vertices. Hence, C' does not belong to the lower envelope of constraints $\{C_2\}$ and does not cross the domain $D_{\text{cone}}(S)$. Then, $C_2(v)$ does not cross the domain too.

Considering the constraint $C_1(v)$, similar arguments are used with projection onto the lower leaning plane. Finally, if all voxels of S are such that the vertical projection of such points lies inside both leaning polygon projections, the voxel v does not change the preimage and thus $D_{3D}(S) = D_{\text{cone}}(S)$. □

In the following we prove that for a digital plane containing at least three leaning points on each line along the y axis or the x axis, $D_{3D}(S)$ does not contain more faces than those described in Proposition 6. In order to prove this statement, we use the Lemma 4 on digital line preimage presented in Section 2 and the following decomposition of a digital plane into digital lines.

Proposition 9. *Let $S \subset P(a, b, c, \mu)$ be a naive plane. Let us define the decomposition of S into* 3D *digital straight lines along the y axis*: $S_j = \{(x, y, z) \in S \mid y = j\}$. *Then we have*

$$S = \bigcup_j S_j \quad \text{and} \quad D_{3D}(S) = \bigcap_j D_{3D}(S_j). \tag{7}$$

We can map each set of voxels S_j to a digital naive line $\text{Proj}(S_j)$ in the (Oxz) plane. This mapping is one-to-one and onto, and, if we denote by $P(a, b, c, \mu)$ the digital naive plane, the digital line is exactly $N(a, -c, \mu - bj)$ (see [2]). In the general case, i.e. on finite subset S of $P(a, b, c, \mu)$, Lemma 10 describes the relation between the parameters of S and those of $\text{Proj}(S_j)$. In the parameter space, the preimage of S_j is a prism whose basis (for $\beta = 0$ and $\beta = 1$) is the preimage of $N(a, -c, \mu - bj)$ and whose directional vector is $(0, 1, -j)^{\mathrm{T}}$. Fig. 8 shows an example of a 3D line S_j preimage and Fig. 9 illustrates the digital plane preimage computation based on the $\{S_j\}$ preimage intersections.

Lemma 10. *Let S be a piece of digital plane of parameters (a, b, c, μ) and S_j a line of its decomposition. Consider a voxel v of S which belongs to S_j, and its projection* $\text{Proj}(v)$ *onto the (Oxz) plane. Then v is a leaning point of S if and only if* $\text{Proj}(v)$ *is a leaning point for the parameters $(a, -c, \mu - bj)$. Moreover, if S_j contains at least* 3 *leaning points of S,* $\text{Proj}(S_j)$ *is the digital line $N(a, -c, \mu - bj)$.*

Proof. Consider a point $v(x, y, z)$ of S. Then we have $ax + by + cz = \mu + r$, with $0 \leqslant r < c$. If v belongs to S_j, then we have $ax + cz = \mu - bj + r$. The point $\text{Proj}(v) = (x, z)$is a leaning point for the parameters $(a, -c, \mu - bj)$ if and only if $ax + cz = \mu - bj + c - 1$ or $ax + cz = \mu - bj$, i.e. if and only if v is a leaning point of S.

Then, if S_j contains at least three leaning points of S, $\text{Proj}(S_j)$ contains three leaning points for the parameters $(a, -c, \mu - bj)$, and then $\text{Proj}(S_j)$ is the digital line $N(a, -c, \mu - bj)$. □

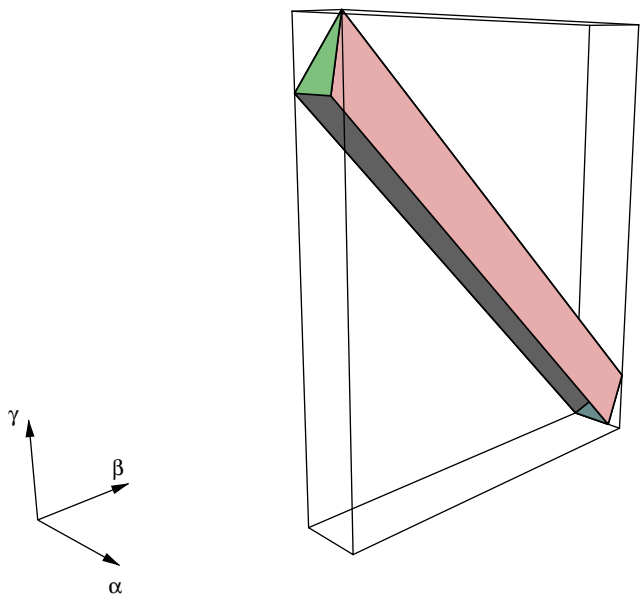

Fig. 8. Preimage of the 3D digital straight line defined by $y = 1$ in the plane $P(1, 3, 4, 0)$.

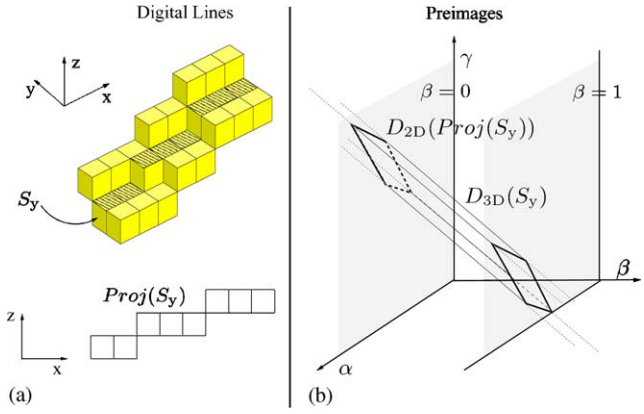

Fig. 9. Illustration of the proof of Theorem 10.

Theorem 11. *Let $S \subset P(a, b, c, \mu)$ a piece of discrete naive plane such that $S = \bigcup_j S_j$ with $S_j = \{(x, y, z) \in S \mid y = j\}$. We assume that for all j, S_j is connected. Then, if each S_j contains at least three leaning points* (*one lower leaning point, one upper leaning point and any third one*)*, we have $D_{3D}(S) = D_{\text{cone}}(S)$.*

Proof. From Proposition 6, the preimage of S has two particular vertices and the faces adjacent to those points correspond to the plane leaning points. Let us consider a voxel

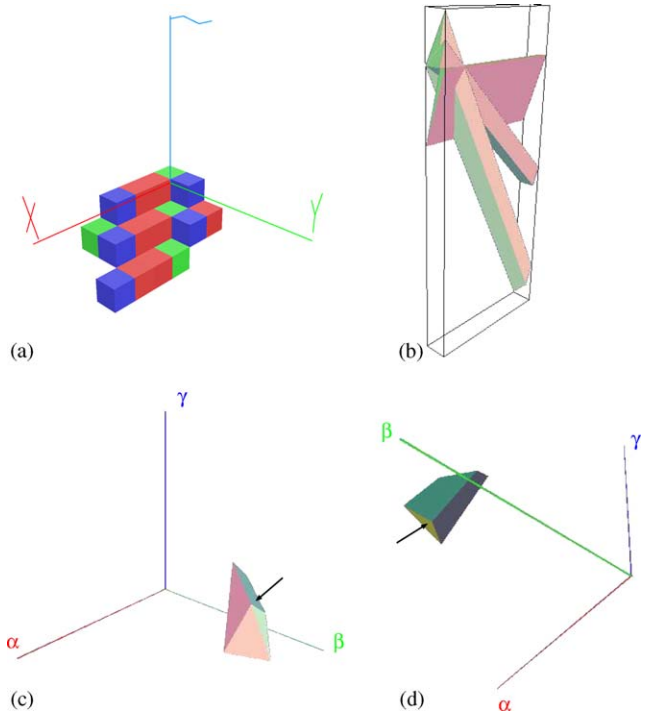

Fig. 10. (a) A piece of plane $P(1, 3, 4, 0)$; (b) preimages of 3D digital straight lines in parameter space (α, β, γ); (c)–(d) the preimage of the piece of plane is the intersection of the digital lines preimages, arrows aim at L^*_{up} (c) and L^*_{low} (d).

$v(x, y, z)$ connected to S which is not a leaning point of S. We show that the preimage of $S \cup v$ is equal to the preimage of S, which proves that any face of $D_{3D}(S)$ goes through either L^*_{up} or L^*_{low}.

Let us consider the decomposition of S into 3D digital lines $\{S_j\}$. Fig. 10 gives an illustration of the notations used in this proof. v belongs to one and only one line of this decomposition, S_y. Thus, only the preimage of S_y may be reduced by v. Now we consider the projection of this line in the 2D space (Oxz), and denote it $\text{Proj}(S_y)$. Hence, the preimage of S_y is a prism whose basis is the preimage of $\text{Proj}(S_y)$. The whole prism is modified by a constraint associated to v if and only if its basis is modified. From Lemma 10 we know that the leaning points of $\text{Proj}(S_y)$ are exactly those of S_y. As a consequence, v is not a leaning point of $\text{Proj}(S_y)$, and according to Lemma 4, the preimage of $\text{Proj}(S_y)$ (denoted $D_{2D}(\text{Proj}(S_y))$ in Fig. 10) does not change after the insertion of the voxel v. Hence, the preimage of S_y does not change either. This means that v does not modify the preimage S_y.

Finally, since $D_{3D}(S)$ is the intersection of all $\{S_j\}$ preimages and since v does not modify the S_y prism, v does not change the domain $D_{3D}(S)$. □

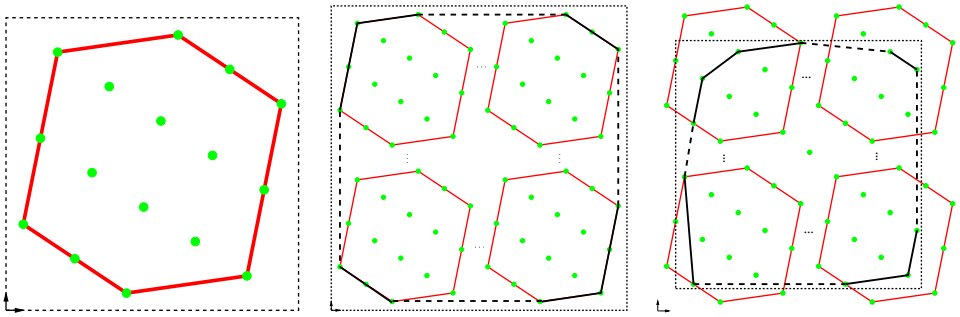

Fig. 11. (a) Set of points generated by $\{(i, gi \bmod h)\}$ with $g = 5$ and $h = 17$, (b) convex hull computation on a $[1, k.h - 1] \times [1, l.h - 1]$ window, (c) convex hull computation on a general window.

4. Bounds on the number of faces

It has been proved in previous theorems how to construct the preimage of a digital plane. In this section, we present a bound on the number of faces of this preimage. Let us suppose a digital plane $S \subset P(a, b, c, \mu)$ satisfying hypothesis of Theorems either 8 or 11 (or both). Hence, the number of faces of D_{3D} is exactly the number of both leaning polygons' vertices. We also suppose that S is a rectangular piece of digital plane. More precisely, there is a one-to-one and onto mapping between the voxels of S and the points in the (Oxy) plane contained in a $[1, m] \times [1, n]$ window.

As given in Definition 5, an arithmetical plane $P(a, b, c, \mu)$ is composed of a set of arithmetical nets given by the solutions of the diophantine equation $ax + by + cz = r$ with $r \in \{\mu, \ldots, \mu + \max(|a|, |b|, |c|) - 1\}$. Given a piece of digital plane, the problem is to bound the number of vertices of the upper (resp. lower) leaning point convex hull. First note that the upper (resp. lower) leaning net can be projected onto the (Oxy) plane without changing the number of vertices of the convex hull. The problem is to consider the convex hull size of the bidimensional net $ax + by = r \pmod c$ in a $m \times n$ window. We first construct two vectors, denoted $U(p, q)$ and $V(s, t)$, that compose a basis of the net using the classical Blankinship's algorithm in number theory [1]. In other words, all upper leaning points are generated by these two vectors. Using scale changes on the grid axis, we can construct a net defined by the canonical vectors $(0, h)$ and $(1, g)$. This one-to-one and onto mapping from the net generated by $[U, V]$ to the net generated by $[(0, h), (1, g)]$ does not change the number of convex hull vertices (given two vectors in the plane, the transformation does not change the sign of the determinant of those vectors).

The net generated by $[(0, h), (1, g)]$ in an $h \times h$ window (see Fig. 11(a)), is exactly the points $\{(i, gi \bmod h)\}$ with $0 < i < h$. As proved by Reveillès and Yaacoub [13], the number of vertices of the convex hull of such points is $O(\log(g))$ (authors illustrate links between such a net and continued fraction of g/h). Hence, over a square $[1, h - 1] \times [1, h - 1]$, the complexity is known (see Fig. 11(a)). In the following, the net in $[1, h - 1] \times [1, h - 1]$ is called a *period*.

The general problem concerns a window W of general size $[1, m] \times [1, n]$. First of all, if W is such that $m = k.h - 1$ and $n = l.h - 1$ with $k,\ l$ in $\mathbb{Z}$, the convex hull of the net can be decomposed into the following elements: two vertical straight segments defined by the horizontal extremal points of the net, two horizontal straight segments joining the vertical extremal points and four parts of the convex hull in one period (see Fig. 11(b)). Hence, the number of vertices of the convex hull over W is bounded by the number of vertices in one period, i.e. $O(\log(g))$.

If we consider a general window W, similar elements are present: we have four straight segments to connect horizontal and vertical extremal points, and four parts of convex hulls of net subsets over a period (see Fig. 11(c)). Hence, we have to study the size of the convex hull on windows $[1, \varepsilon] \times [1, \varepsilon']$ with $\varepsilon,\ \varepsilon'$ in $\mathbb{Z}$ and $\varepsilon, \varepsilon' < h$. In fact, Reveillès and Yaacoub also proved that the convex hull size of the net $\{(i, (gi + \varepsilon \bmod h)\}$ with ε in $\{0, \ldots, h\}$, over a period is also bounded by $O(\log(g))$ [13]. In other words, we can shift the net by the vector $(0, \varepsilon)$, we do not change the size of the convex hull. Similarly, we can prove that the translation by the vector $(\varepsilon', 0)$ do not change the complexity of the convex hull too. Hence, the number of vertices of a piece of the convex hull in a window $[1, \varepsilon] \times [1, \varepsilon']$ is always bounded by $O(\log(g))$. Finally, the size of the overall convex hull in the window W is bounded by $O(\log(g))$.

Hence, given a $m \times n$ rectangular digital plane with minimal parameters (a, b, c, μ), we first have $0 \leqslant a \leqslant b < c \leqslant \max(m, n)$. Since, g and h are linear combinations of a, b and c, the size of both the upper and the lower leaning polygon is bounded by $O(\log(\max(m, n)))$.

Theorem 12. *Let S be a rectangular $m \times n$ piece of digital plane satisfying hypothesis of either Theorem* 8 *or* 11 (*or both*). *Then the number of faces of the preimage of S is bounded by* $O(\log(\max(m, n)))$.

5. Conclusion

In this paper we have presented some first results about digital plane preimage. We have shown that with some hypothesis on the piece of digital plane, the shape of the preimage is a double cone whose structure is very similar to the one of 2D digital straight segments preimages.

We have also introduced the decomposition of a digital plane segment into 3D digital straight segments, which suggests interesting arithmetical properties on the polyhedron's faces and vertices. Indeed, each 3D digital line segment preimage face is resulting from a side of a 2D digital segment preimage which have known arithmetical structure.

Finally, we have shown that under some hypotheses, the number of faces of a rectangular $m \times n$ digital plane segment preimage is bounded by $O(\log(\max(m, n)))$. As in 2D, such a result together with the other ones of this paper can lead to the design of a very efficient digital plane recognition algorithm. In future works, efforts must be made to enlarge the scope of the theorems whole considering general pieces of plane without constraints.

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