Digital geometry fundaments : Application to plane recognition

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1 Introduction

Triangulation, quadrangulation problems and more generally 3D objects polyhedrization are an important subject of research. In digital geometry, a 3D object is seen as a set of voxels placed in a representation space only constituted of integers. The objective of the polyhedrization is to obtain a complete description of the object with faces, edges and vertices. The recognition of digital planes is a first step which is very important. We will focus on digital naive planes that have been studied through their configurations of tricubes [Sch97, VC97], of (n, m)-cubes [VC99b] and connected or not connected voxels set [VC99a, Gér99]. The link between the normal equation of a plane and configuration of voxels set has been studied by the construction of the corresponding Farey net [VC99a]. We can find many references about the recognition of digital planes. Some algorithms were related to the construction of the convex hull of the studied voxels set [KS91, KR82]. Other approaches use linear programming [ST91], mean square approximation [BF94] or Fourier-Motzkin transform [FP99, FST96, Vee94]. The first algorithms entirely discrete were to recognize rectangular pieces of naive planes [Deb95, DRR94, VC99b]. In this paper, we describe an incremental algorithm to recognize any coplanar voxels set as a digital naive plane by using Farey nets. Then we propose a polyhedrization method able to give all the digital naive planes of the surface of the 3D object.

2 Definitions

Let a, b, c, r be four integers such as a, b, c are not null all together and verify $\gcd(a, b, c) = 1$.
The digital naive plane $\mathcal{P}(a, b, c, r)$, where (a, b, c) is its normal vector and r its translation parameter, is the set of points (x, y, z) in $\mathbb{Z}^3$ verifying:

$$0 \leq ax + by + cz + r < \max(|a|, |b|, |c|)$$

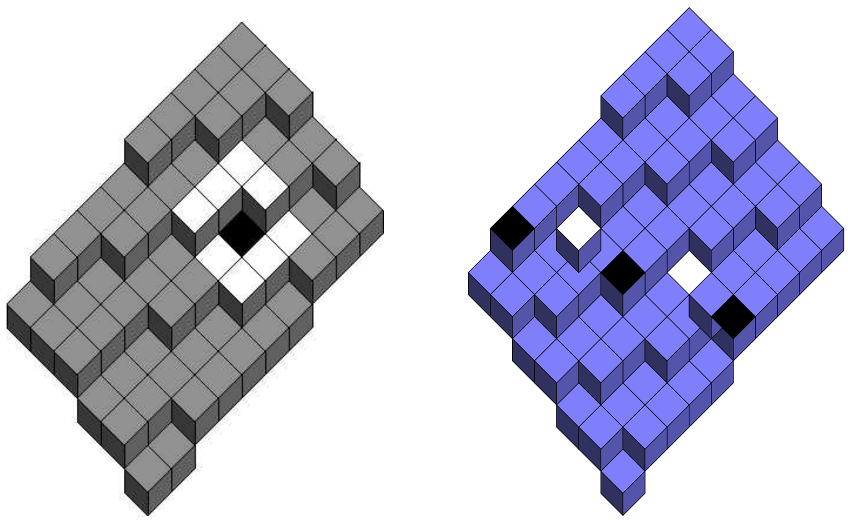
We will limit our study to naive planes $\mathcal{P}(a, b, c, r)$ in the 48th part of space such as $0 \leq a \leq b \leq c$ and $c \neq 0$. These planes are `functional` in $0xy$. For each point (x, y) in $\mathbb{Z}^2$, we have only one point (x, y, z) in $\mathbb{Z}^3$ belonging to the naive plane.

Let us notice $f(a,b,c,r)$ the function from $\mathbb{Z}^2$ to $\mathbb{Z}$ defined by:

$$f(a,b,c,r)(u,v) = -\left[\frac{au+bv+r}{c}\right]$$

where $[w]$ denotes the integer part of the real number w, then $z = f(a,b,c,r)(x,y)$.

The points (x,y,z) of the naive plane $\mathcal{P}(a,b,c,r)$ which verify $ax+by+cz+r=0$ (resp. $ax+by+cz+r=c-1$) are the `lower` (resp. `upper`) `leaning points` of the naive plane.



(a) (b)

Fig. 1. (a) A naive plane represented as a set of voxels : a voxel (black) surrounded by its eight neighbors (white) (b) A naive plane with the lower leaning points (white voxels) and upper leaning points (black voxels).

3 Equivalence class of a voxels set

Let n in $\mathbb{N}^*$ and $V = \{(i_1,j_1),\cdots,(i_n,j_n)\}$ a set of n points of $\mathbb{Z}^2$. The cluster of voxels $S(a,b,c,r)(x,y)$ of the naive plane $\mathcal{P}(a,b,c,r)$ indexed by V and with origin (x,y) is defined by:

$$S(a,b,c,r)(x,y) = \bigcup_{q=1}^{n}\{(x+i_q,y+j_q,z+k_q) \mid (i_q,j_q)\in V,\ k_q = f(a,b,c,r)(x+i_q,y+j_q)-z\}$$

where $z = f(a,b,c,r)(x,y)$.

$S(a,b,c,r)(x,y)$ is the part of the naive plane going through the n voxels $(x + i_q, y + j_q, z + k_q)$, $q = 1, \cdots, n$.

Let a point (x,y,z) of the naive plane $\mathcal{P}(a,b,c,r)$. It verifies :

$$0 \leq ax + by + cz + r < c$$

Let (x_l, y_l, z_l) be a lower leaning point from $\mathcal{P}(a,b,c,r)$. It belongs to the naive plane, so :

$$ax_l + by_l + cz_l + r = 0$$

The point (x,y,z) can be written as $(x_l + u, y_l + v, z_l + w)$ with (u,v,w) in $\mathbb{Z}^3$. The double inequality becomes :

$$0 \leq a(x_l + u) + b(y_l + v) + c(z_l + w) + r < c$$

So, we obtain :

$$0 \leq au + bv + cw < c$$

Let us notice r' equal to $au + bv + cw$. For $q = 1, \cdots, n$, the point $(x + i_q, y + j_q, z + k_q)$ verifies :

$$0 \leq a(x + i_q) + b(y + j_q) + c(z + k_q) + r < c$$

$$0 \leq a(x_l + u + i_q) + b(y_l + v + j_q) + c(z_l + w + k_q) + r < c$$

$$0 \leq a(u + i_q) + b(v + j_q) + c(w + k_q) + r < c$$

$$0 \leq ai_q + bj_q + ck_q + r' < c$$

Consequently, $S(a,b,c,r)(x,y)$ can be written as:

$$S(a,b,c,r)(x,y) = \{(x,y,z)\} \oplus S(a,b,c,r')(0,0)$$

where $A \oplus B$ is the Minkowski sum between the sets A and B.

All real planes for which the discretization by the *object boundary quantization* method on the set $\{(x,y)\} \oplus V$ is the set $S(a,b,c,r)(x,y)$, have to go through the point (x_l, y_l, z_l). Moreover on the point (x,y) the discretization must be the voxel (x,y,z). A first equivalence class of $S(a,b,c,r)(x,y)$ is the set of parameters (α, β) with $0 \leq \alpha \leq \beta \leq 1$ of the real plane $\alpha(x - x_l) + \beta(y - y_l) + z - z_l = 0$ verifying $0 \leq \alpha u + \beta v + w < 1$. We notice this equivalence class by the set:

$$\overline{S}(r)(x,y) = \bigcap_{q=1}^{n} \{(\alpha,\beta) \mid 0 \leq \alpha \leq \beta \leq 1,$$
$$(x + i_q, y + j_q, z + k_q) \in S(a,b,c,r)(x,y), -1 < i_q\alpha + j_q\beta + k_q < 1\}$$

For each integer point (i,j), we have $f(a,b,c,r)(x_l+i,y_l+j) = f(a,b,c,r)(x_l,y_l) + f(a,b,c,0)(i,j)$. For $q = 1,\cdots,n$, the integer k_q satisfies $k_q = f(a,b,c,0)(u + i_q, v + j_q) - f(a,b,c,0)(u,v)$. So $\overline{S}(r)(x,y)$ is equal to $\overline{S}(0)(u,v)$.

Let $\mathcal{D}(i,j,k)$ be the line in the parametric space $W = \{(\alpha,\beta),\ 0 \le \alpha \le \beta \le 1\}$ with equation $i\alpha + j\beta + k = 0$. Let $B(i,j,k)$ be the open-band of this space limited by the two parallel lines $\mathcal{D}(i,j,k+1)$ and $\mathcal{D}(i,j,k-1)$.
The equivalence class becomes:

$$\overline{S}(r)(x,y) = W \bigcap \left(\bigcap_{q=1}^{n} B(i_q, j_q, k_q) \right)$$

In a previous work [VC99a], we proved that the voxels set $\mathcal{E}$ centered on the lower leaning point (x_l, y_l, z_l) and defined by :

$$\mathcal{E} = \bigcup_{q=1}^{n} S(a,b,c,r)(x_l - i_q, y_l - j_q)$$

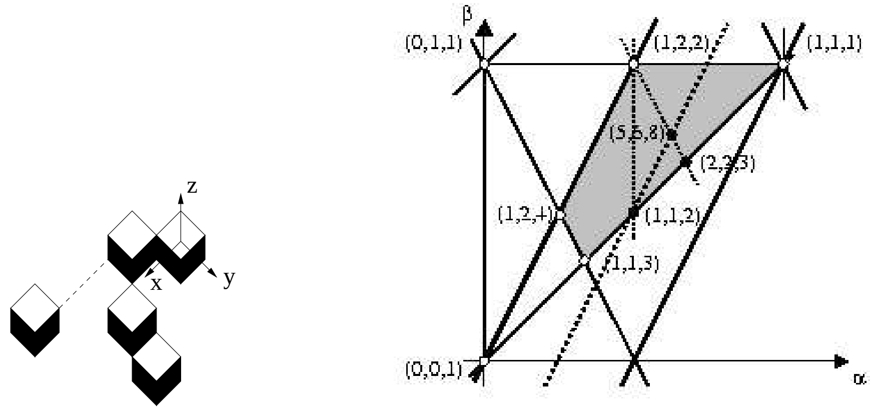
is a complete system. It is representative of the different configurations of voxels sets defined on V which generate the naive planes with normal (a,b,c). The equivalence class $\overline{\mathcal{E}}$ of that set is the intersection of the open bands $B(i,j,k)$ for (i,j) belonging to $V \ominus V$ ($\ominus$ designs the Minkowski difference between two sets) and k verifying $k = f(a,b,c,r)(x_l+i, y_l+j) - f(a,b,c,r)(x_l,y_l)$. The equivalence classes of the different configurations appearing around leaning points split the space W in polygonal areas called *Farey net associated to voxels sets defined on* V.

Example 1. Let A be the voxels set of a naive plane defined on $V = \{(0,0), (1,-1), (2,0), (2,1), (4,-2)\}$ and illustrated in figure 2(a). The equivalence class $\overline{A}$ of that set is the intersection of the five bands $B(0,0,0)$, $B(1,-1,0)$, $B(2,0,-1)$, $B(2,1,-2)$ and $B(4,-2,-1)$ (cf. Fig. 2(b)). Each rational point in that area corresponds to the parameters of a naive plane containing that configuration of voxels set.

Now if we look for real planes $\alpha x + \beta y + z + \gamma = 0$ with $0 \le \alpha \le \beta \le 1$ for which the discretization by the object boundary quantization method on the point (x,y) is the voxel (x,y,z) then the parameter γ has to verify $\gamma = \gamma' - (\alpha x + \beta y + z)$ with $0 \le \gamma' < 1$. Moreover if the discretization on $\{(x,y)\} \oplus V$ is the set $S(a,b,c,r)(x,y)$, the parameters of that planes belong to the set:

$$\overline{S}'(x,y) = \bigcap_{q=1}^{n} \{(\alpha, \beta, \gamma' - (\alpha x + \beta y + z)) \mid 0 \le \alpha \le \beta \le 1,\ 0 \le \gamma' < 1,$$
$$(x + i_q, y + j_q, z + k_q) \in S(a,b,c,r)(x,y),\ 0 \le i_q\alpha + j_q\beta + \gamma' + k_q < 1\}$$

This second equivalence class will be used in the recognition algorithm.



(a) (b)

Fig. 2. (a) Set $A = \{(0,0,0),(1,-1,0),(2,0,-1),(2,1,-2)\,(4,-2,-1)\}$; (b) Equivalence class of A.

4 Recognition algorithm

Let $S = \{(x_q, y_q, z_q),\ q = 1, \cdots, n\}$ be a set of n voxels.
We are going to establish an incremental algorithm to identify the parameters of the naive planes going through the n points of S.

The naive planes solutions are the planes $\mathcal{P}(a, b, c, r - (ax_1 + by_1 + cz_1))$ for which the parameters $\left(\frac{a}{c}, \frac{b}{c}, \frac{r}{c}\right)$ belong to the set:

$$\overline{S} = \left\{(\alpha, \beta, \gamma) \in [0,1]^2 \times [0,1[\,\mid\, \forall q \in \{1, \cdots, n\} \quad 0 \le i_q\alpha + j_q\beta + \gamma + k_q < 1\right\}$$

where (i_q, j_q, k_q) is defined as to be the integer points $(x_q - x_1, y_q - y_1, z_q - z_1)$ for $q = 1, \cdots, n$.

Let $\mathcal{B}_q$ in $\mathbb{N}^4$ be the set of vectors (a, b, c, r) such that the projection in the plane $(c = 1)$ are the vertices of the convex hull of the space containing the parameters (α, β, γ) of real planes for which the discretization on the point (i_p, j_p) is the point (i_p, j_p, k_p) for p varying from 1 to q.

We are going to construct the sets $\mathcal{B}_q$ for $q = 1, \cdots, n$. The following algorithm gives at step q the set $\mathcal{B}_q$ or the empty set if there is no solution. In the first case, the solution with the minimal periodicity can correspond to a vertex of the convex-hull, the median point between the projection of two vertices of $\mathcal{B}_q$ or the median point of the area limited by the projection of the vectors of $\mathcal{B}_q$.
As the discretization of all real planes of the working space goes through the

origin (here the origin is taken at point (x_1, y_1, z_1)), the algorithm starts with

$$\mathcal{B}_1 = \{(0,0,1,0),(0,1,1,0),(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

composed by the six vectors (a, b, c, r) such that the projection in the plane $(c = 1)$ are the vertices of the convex-hull limiting the solution space of (α, β, γ).

Algorithm at step q, $q \geq 2$:
We introduce the point (i_q, j_q, k_q).
Let L_q and L_q^+ the functions from $\mathbb{N}^4$ to $\mathbb{Z}$ defined by:

$$\begin{aligned} L_q(a,b,c,r) &= ai_q + bj_q + ck_q + r \\ L_q^+(a,b,c,r) &= L_q(a,b,c,r) - c \end{aligned}$$

The naive plane of parameters (a, b, c, r) goes through the voxel (i_q, j_q, k_q) if and only if $0 \leq L_q(a, b, r, c) < c$. Consequently the vectors (a, b, c, r) of $\mathcal{B}_q$ verify $L_p(a, b, c, r) \geq 0$ and $L_p^+(a, b, c, r) \leq 0$ for $p = 1, \cdots, q$.

Initialization: $\mathcal{B}_q = \emptyset$.

For all the vectors V_i in $\mathcal{B}_{q-1}$, $i = 1, \cdots, \#(\mathcal{B}_{q-1})$ do:

1. Process

   Step 1 : If $L_q(V_i) \geq 0$ and $L_q^+(V_i) \leq 0$ then the projection of V_i is still on the convex hull of the domain solution. We insert V_i in $\mathcal{B}_q$. More particularly, if $L_q(V_i) = 0$ (resp. $L_q^+(V_i) = 0$) we can say that the voxel (i_q, j_q, k_q) (resp. $(i_q, j_q, k_q - 1)$) is a lower leaning point of the naive plane of parameters V_i.

   Step 2 : If $L_q(V_i) < 0$ (resp. $L_q^+(V_i) > 0$), we are going to search the point P such as $L_q(P) = 0$ (resp. $L_q^+(P) = 0$). To do that, we use an algorithm based on the notion of median point [Far16, Gra92].

   For each vector V_j, $j > i$, belonging to $\mathcal{B}_{q-1}$ and verifying $L_q(V_j) > 0$ (resp. $L_q^+(V_j) < 0$) **do**
       $P_1 = V_i$ and $P_2 = V_j$
       While $L_q(P_1) + L_q(P_2) \neq 0$ (resp. $L_q^+(P_1) + L_q^+(P_2) \neq 0$) **do**
           if $L_q(P_1) + L_q(P_2)$ and $L_q(V_i)$ (resp. $L_q^+(P_1) + L_q^+(P_2)$ and $L_q^+(V_i)$) have the same sign **then**
               $P_1 = P_1 + P_2$ and $P_2 = V_j$
           else
               $P_1 = V_i$ and $P_2 = P_1 + P_2$
       End While
   End For
   The solution is given by $P = P_1 + P_2$.

The point (i_q, j_q, k_q) (resp. $(i_q, j_q, k_q - 1)$) is a lower leaning point of the naive plane of parameters P.
We insert the point P in $\mathcal{B}_q$.

2. Validation of $\mathcal{B}_q$

   (a) For each vector $V = (a, b, c, r)$ in $\mathcal{B}_q$ we verify if the projection in the plane $(c = 1)$ is a vertex of the convex hull. If V can be written as a combination of vectors of $\mathcal{B}_q$ then the projection of V is on the convex hull but it is not a vertex. So we suppress that point from the list.
   (b) If $\#(\mathcal{B}_q) \leq 2$, there is no solution and we suppress all the vertices from the list.
   (c) If $\#(\mathcal{B}_q) = 3$, we verify that the points $(a/c, b/c)$ corresponding to the vectors (a, b, c, r) of $\mathcal{B}_q$ are not alined otherwise we suppress the vertices from the list.

Example 2. We are going to illustrate this algorithm on an example. We want to know if the set of voxels in figure 3 belong to a naive plane of the studied 48th part of space. We start with a first point defined as the origin of the voxels

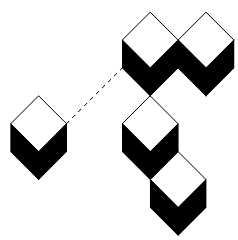


Fig. 3. Set of voxels to recognize.

set (cf. Fig 4(a)). The parameters set of naive planes including the origin are the rational points (α, β, γ) contained in the domain limited by the projection $(a/c, b/c, r/c)$ of the vectors (a, b, c, r) of $\mathcal{B}_1$. The set $\mathcal{B}_1$ is composed by the six vectors:

$$\mathcal{B}_1 = \{(0, 0, 1, 0), (0, 1, 1, 0), (1, 1, 1, 0), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$$

As it was previously mentioned, it is equivalent to say that parameters (α, β) belong to the area limited by the points $(a/c, b/c)$ (cf. Fig 4(b)). We introduce the second point $(1, -1, 0)$ (cf. Fig 5(a)). Let us compute the value $L_2(V)$ and $L_2^+(V)$ on the different vectors V from $\mathcal{B}_1$:

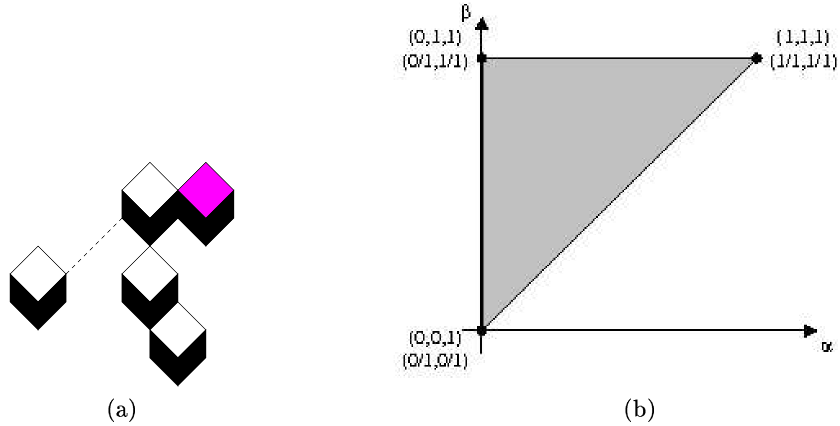


Fig. 4. (a) Set $\{(0,0,0)\}$; (b) Equivalence class.

(a,b,c,r)	$L_2(a,b,c,r) = a-b+r$	$L_2^+(a,b,c,r) = a-b+r-c$
$(0,0,1,0)$	0	-1
$(0,1,1,0)$	-1	-2
$(1,1,1,0)$	0	-1
$(0,0,1,1)$	1	0
$(0,1,1,1)$	0	-1
$(1,1,1,1)$	1	0

The vectors $(0,0,1,0)$, $(1,1,1,0)$, $(0,0,1,1)$, $(0,1,1,1)$ and $(1,1,1,1)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_2$. As $L_2(0,1,1,0) < 0$, we apply the algorithm presented in step 2. We introduce the new vectors $(0,1,2,1)$ and $(1,2,2,1)$. But $(0,1,2,1)$ is the sum of vectors $(0,0,1,0)$ and $(0,1,1,1)$. Similarly, the vector $(1,2,2,1)$ is the sum of vectors $(0,1,1,1)$ and $(1,1,1,0)$. Consequently, the vectors $(0,1,2,1)$ and $(1,2,2,1)$ are not present in $\mathcal{B}_2$.
Finally, we have:

$$\mathcal{B}_2 = \{(0,0,1,0),(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a',b',c',r') belonging to $\mathcal{B}_2$ are solutions (cf. Fig 5(b)). We introduce the third point $(2,0,-1)$ (cf. Fig 6(a)). We compute the value $L_3(V)$ and $L_3^+(V)$ on the different vectors V of $\mathcal{B}_2$:

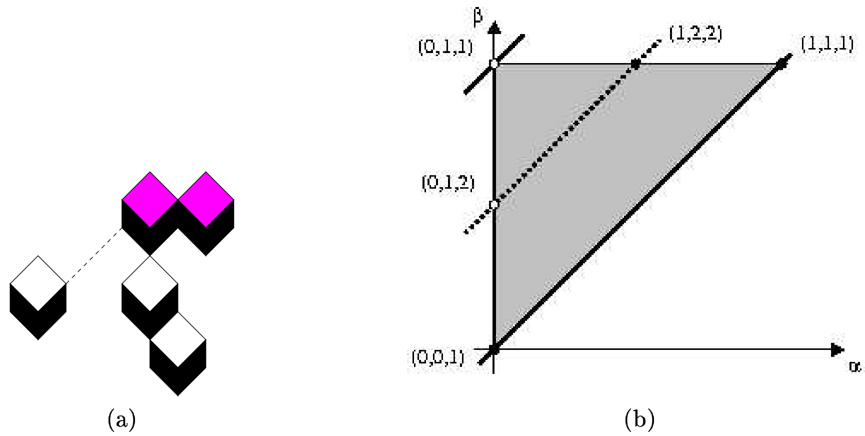


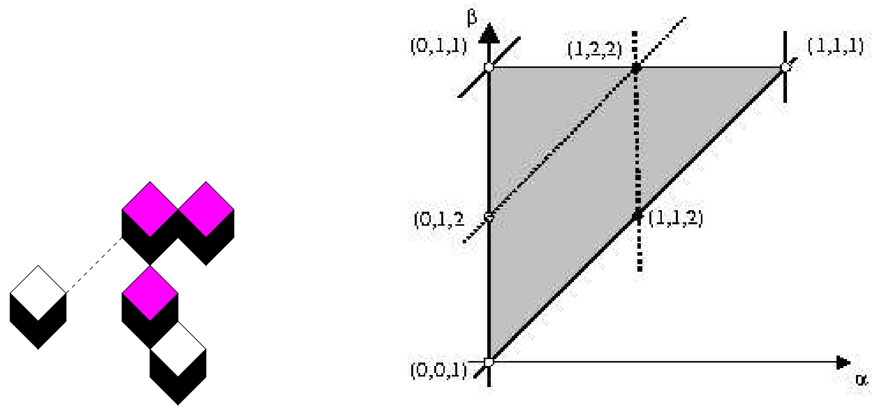
Fig. 5. (a) Set $\{(0,0,0),(1,-1,0)\}$; (b) Equivalence class.

(a,b,c,r)	$L_3(a,b,c,r)=2a-c+r$	$L_3^+(a,b,c,r)=2a-2c+r$
$(0,0,1,0)$	-1	-2
$(1,1,1,0)$	1	0
$(0,0,1,1)$	0	-1
$(0,1,1,1)$	0	-1
$(1,1,1,1)$	2	1

The vectors $(1,1,1,0)$, $(0,0,1,1)$ and $(0,1,1,1)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_3$. As $L_3(0,0,1,0)<0$ and $L_3^+(1,1,1,1)>0$, we apply for these vectors the algorithm presented in step 2. We make appear the new vectors $(1,1,2,0)$, $(1,1,3,1)$, $(2,2,3,2)$, $(1,1,2,2)$ and $(1,2,2,2)$. But we have: $(1,1,3,1)=(1,1,2,2)+(1,1,1,0)$ and $(2,2,3,2)=(1,1,2,2)+(1,1,1,0)$. These two vectors are not inserted.
Finally, we have:

$$\mathcal{B}_3=\{(1,1,1,0),(0,0,1,1),(0,1,1,1),(1,1,2,0),(1,1,2,2),(1,2,2,2)\}$$

Every naive planes for which the projection $(a/c,b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c',b'/c')$ with (a',b',c',r') belonging to $\mathcal{B}_3$ are solutions (cf. Fig 6(b)). More particularly, the naive planes with normal $(1,1,2)$ or $(1,2,2)$ contains the initial configuration of 3 voxels. We introduce the fourth point $(2,1,-2)$ (cf. Fig 7(a)). We compute the value $L_4(V)$ and $L_4^+(V)$ on the different vectors V of $\mathcal{B}_3$:



(a) (b)

Fig. 6. (a) Set $\{(0,0,0),(1,-1,0),(2,0,-1)\}$; (b) Equivalence class : the two bold lines represent the boundary of the band and the dotted line corresponds to the medial line of the band.

(a,b,c,r)	$L_4(a,b,c,r) = 2a+b-2c+r$	$L_4^+(a,b,c,r) = 2a+b-3c+r$
$(1,1,1,0)$	1	0
$(0,0,1,1)$	-1	-2
$(0,1,1,1)$	0	-1
$(1,1,2,0)$	-1	-3
$(1,1,2,2)$	1	-1
$(1,2,2,2)$	2	0

The vectors $(1,1,1,0)$, $(0,1,1,1)$, $(1,1,2,2)$ and $(1,2,2,2)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_4$. As the value L_4 is negative for the vectors $(0,0,1,1)$ and $(1,1,2,0)$, we applied for these vectors the algorithm presented in step 2. We make appear the new vectors $(1,1,2,1)$, $(1,1,3,3)$, $(2,2,3,0)$, $(1,2,4,4)$ and $(3,4,6,2)$. But we have: $(1,2,4,4) = (1,1,3,3)$ $+(0,1,1,1)$ and $(3,4,6,2) = (1,1,2,1)+(0,1,1,1)+(2,2,3,0)$.These two vectors are not inserted.
Finally, we have:

$$\mathcal{B}_4 = \{(1,1,1,0),(0,1,1,1),(1,1,2,2),(1,2,2,2),(1,1,2,1),(1,1,3,3),(2,2,3,0)\}$$

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a',b',c',r') belonging to $\mathcal{B}_4$ are solutions (cf. Fig 7(b)). More particularly, the naive planes with normal $(1,1,2)$ or $(1,2,2)$ or $(2,2,3)$ contains the set of 4 voxels. We can verify in figure 7(c) that this configuration of voxels set is contained in the naive plane with normal $(1,1,2)$. We introduce the fifth point $(4,-2,-1)$ (cf. Fig 8(a)). We

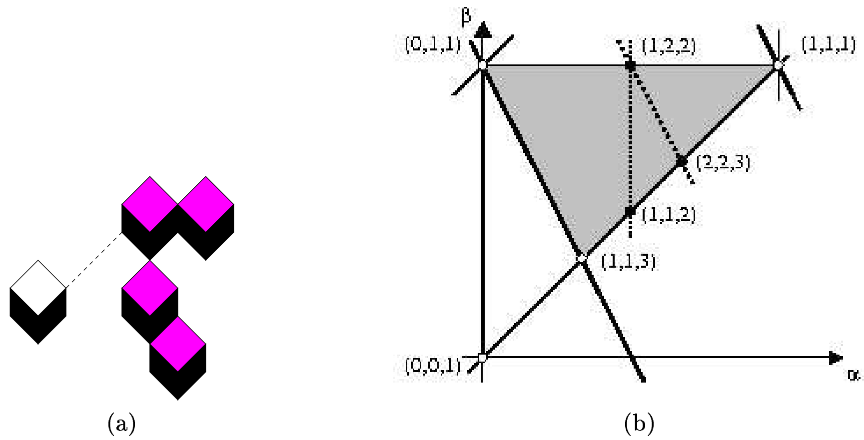


Fig. 7. (a) Set $\{(0,0,0),(1,-1,0),(2,0,-1),(2,1,-2)\}$; (b) Equivalence class.

compute the value $L_5(V)$ and $L_5^+(V)$ on the different vectors V of $\mathcal{B}_5$:

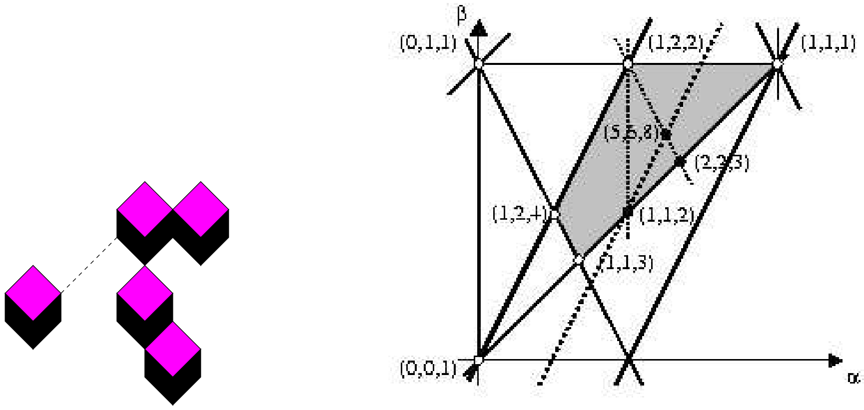
(a,b,c,r)	$L_5(a,b,c,r) = 4a-2b-c+r$	$L_5^+(a,b,c,r) = 4a-2b-2c+r$
$(1,1,1,0)$	1	0
$(0,1,1,1)$	-2	-3
$(1,1,2,2)$	2	0
$(1,2,2,2)$	0	-2
$(1,1,2,1)$	1	-1
$(1,1,3,3)$	2	-1
$(2,2,3,0)$	1	-2

The vectors $(1,1,1,0)$, $(1,1,2,2)$, $(1,2,2,2)$, $(1,1,2,1)$, $(1,1,3,3)$, and $(2,2,3,0)$ verify the property indiced by step 1 of the algorithm. We insert these vectors in $\mathcal{B}_5$. As the value L_5 is negative for the vector $(0,1,1,1)$, we applied for this vector the algorithm presented in step 2. We make appear the new vectors $(2,3,3,1)$, $(1,2,3,3)$, $(2,3,5,3)$, $(1,2,4,4)$ and $(4,5,7,1)$. But we have: $(2,3,3,1) = (1,1,1,0) + (1,1,2,1)$. This vector is not inserted. Finally, we have:

$$\mathcal{B}_5 = \{(1,1,1,0),(1,1,2,2),(1,2,2,2),(1,1,2,1),(1,1,3,3),$$
$$(2,2,3,0),(1,2,3,3),(2,3,5,3),(1,2,4,4),(4,5,7,1)\}$$

Every naive planes for which the projection $(a/c, b/c)$ of the normal (a,b,c) is contained in the area limited by the points $(a'/c', b'/c')$ with (a',b',c',r') belonging to $\mathcal{B}_5$ are solutions (cf. Fig 8(b)).

More particularly, the naive planes with normal $(1,1,2)$ or $(2,2,3)$ contains the set of five voxels. As an example, we can verify that this configuration of voxels



(a) (b)

Fig. 8. (a) Set $\{(0,0,0),(1,-1,0),(2,0,-1),(2,1,-2)\}$; (b) Equivalence class.

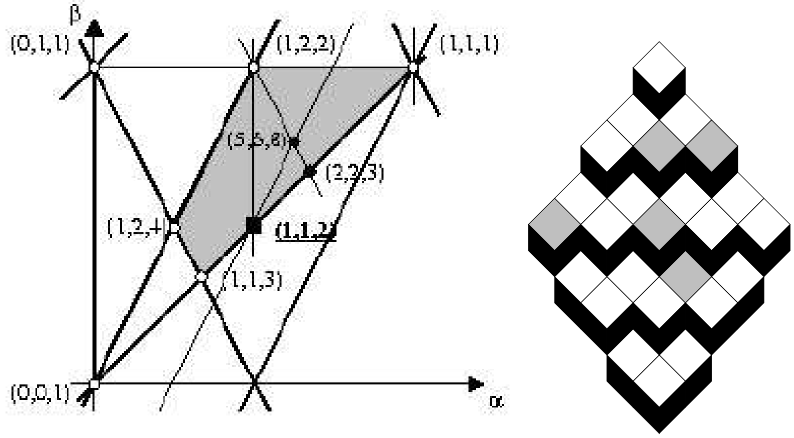
set is contained in the naive planes with normal $(1,1,2)$ (in figure 9) or $(5,7,9)$ (in figure 10).

5 Polyhedrization of a voxels object

The polyhedrization of a voxelized object has been studied using mainly approximation approaches [BF94]. Here we propose an algorithm which fully works on the discrete representation of digital naive planes. The present algorithm is an iterative process based on the recognition algorithm described in the previous part.

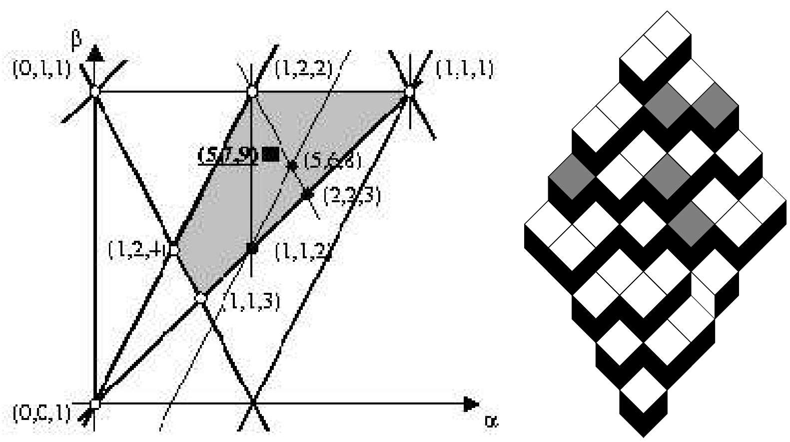
The algorithm treats independently each direction of the 3D object. So, for each direction and as long as surfels are not totally analyzed, a surfel is chosen. Then we verify if the surfel and its eight neighbors are coplanar with the naive plane recognition algorithm. If the voxels belongs to a same plane, a number is given to the nine surfels. The eight neighbors are put into a list of faces waiting to be processed. While this list is not empty, the current naive plane is extended if it is possible with faces into the list and their eight neighbors. The neighbors of the new coplanar surfels are added to the list. When no surfel of the list is coplanar with the voxels of the plane, the list is cleaned and a new surfel is chosen as the origin of a new naive plane and the number of the current plane is incremented. As a result, the naive planes are extended to the maximum.

The obtained results are shown on two images (cf. Fig 11), where each number or grey level corresponds to a different naive plane. The image presented in (a) is a synthetic image: it is a pyramid with four visible faces and another one behind (the base). All planes are well recognized and are according to expectation. The second image presented in (b) as a X-ray scanner image of a hand os



(a) (b)

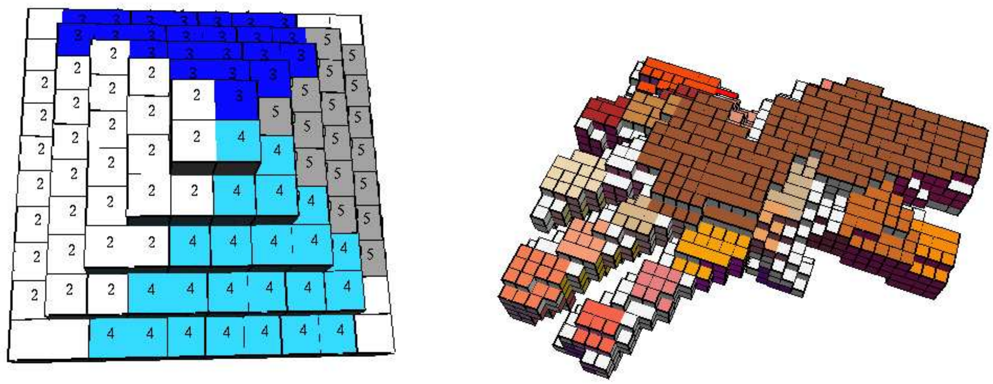
Fig. 9. (a) Farey net : plane with normal $(1, 1, 2)$(black square); (b) Part of the naive plane with normal $(1, 1, 2)$; in dark grey, we recognize the initial configuration of the five voxels.



(a) (b)

Fig. 10. (a) Farey net : plane with normal $(5, 7, 9)$(black square); (b) Part of the naive plane with normal $(5, 7, 9)$; in dark grey, we recognize the initial configuration of the five voxels.

size 30x30x30 voxels. The result show that planes are extended to their maximum even if the image is a difficult one for this algorithm because there is a lot of small planes.



(a) (b)

Fig. 11. (a) Result on a pyramid image with four faces (numbered 2,3,4 and 5) (b) Result on a X-ray scanner image of a hand.

6 Conclusion

A generic algorithm for coplanar voxels recognition has been presented. This algorithm analyzes any configuration of voxels set either connected or not connected. It is fully discrete working in the dual space issued from Farey net representation of the normal equation of a digital naive plane. This algorithm has been used for polyhedrization of the boundary of voxelized objects. Good results have been obtained on synthetic and real images.
As perspective, a lot of work has to be achieved to find the position of the intersections of planes in order to obtain edges and vertices. Then we will have a complete polyhedrization of 3D objects. Many applications in visualization or compression could be studied when the previous point will be solved.

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