Fractal Approximation and Compression using Projected IFS

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Overview

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  - Model overview
- IFS
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- Results
  - Curve approximation
  - Surface approximation
  - Image compression

Introduction

Fractal inverse problem

- NP-hard problem [Ruhl and Hartenstein 1997]
- Direct methods
  - Wavelet [Struzik et al. 1995, Berkner 1997]
  - Complex moment [Abiko et al. 1997]
- Indirect methods
  - Combinatory exploration: fractal compression [Jacquin 1992]
  - Stochastic methods [Lutton et al. 1995, Vences and Rudomin 1994, Goertzel et al. 1994]
- Approximation of a signal property [Véhel and Daoudi 1996]

Introduction (2)

Model overview

- Combining
  - Fractal model: Iterated Function Systems
  - CAGD model: free forms
- *Projected IFS attractors*

- Provide differential properties
- Allows differential iterative methods to operate
  - More general than models used in direct methods
  - Less complex than methods used in indirect methods

IFS

Formalism

- Used to describe fractal objects
- Based on self-similarity
- An object is described by a set of contractions of a metric space E:

$$\mathbb{T} = \{T_0, \ldots, T_N - 1\}$$

- Attractor $\mathcal{A}(\mathbb{T})$ of $\mathbb{T}$:

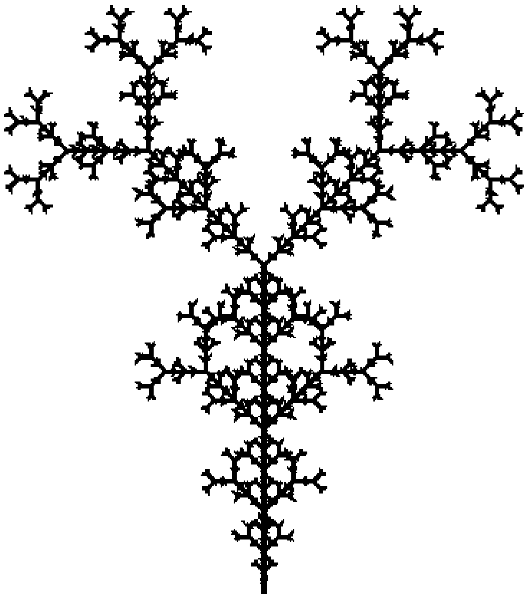
$$\mathcal{A}(\mathbb{T}) = \lim_{n \to \infty} \mathbb{T}^n K$$

where K is a random compact.

IFS (2)

Example

- In $\mathbb{R}^2$



$$
\begin{aligned}
T_0 &= S(0.5) \\
T_1 &= T(0,0.5)S(0.5) \\
T_2 &= T(0,1)R(\pi/4)S(0.5) \\
T_3 &= T(0,1)R(-\pi/4)S(0.5)
\end{aligned}
$$

$$\mathcal{A}(\mathbb{T}) = T_0\mathcal{A}(\mathbb{T}) \cup T_1\mathcal{A}(\mathbb{T}) \cup T_2\mathcal{A}(\mathbb{T}) \cup T_3\mathcal{A}(\mathbb{T})$$

Projected IFS model

Formalism

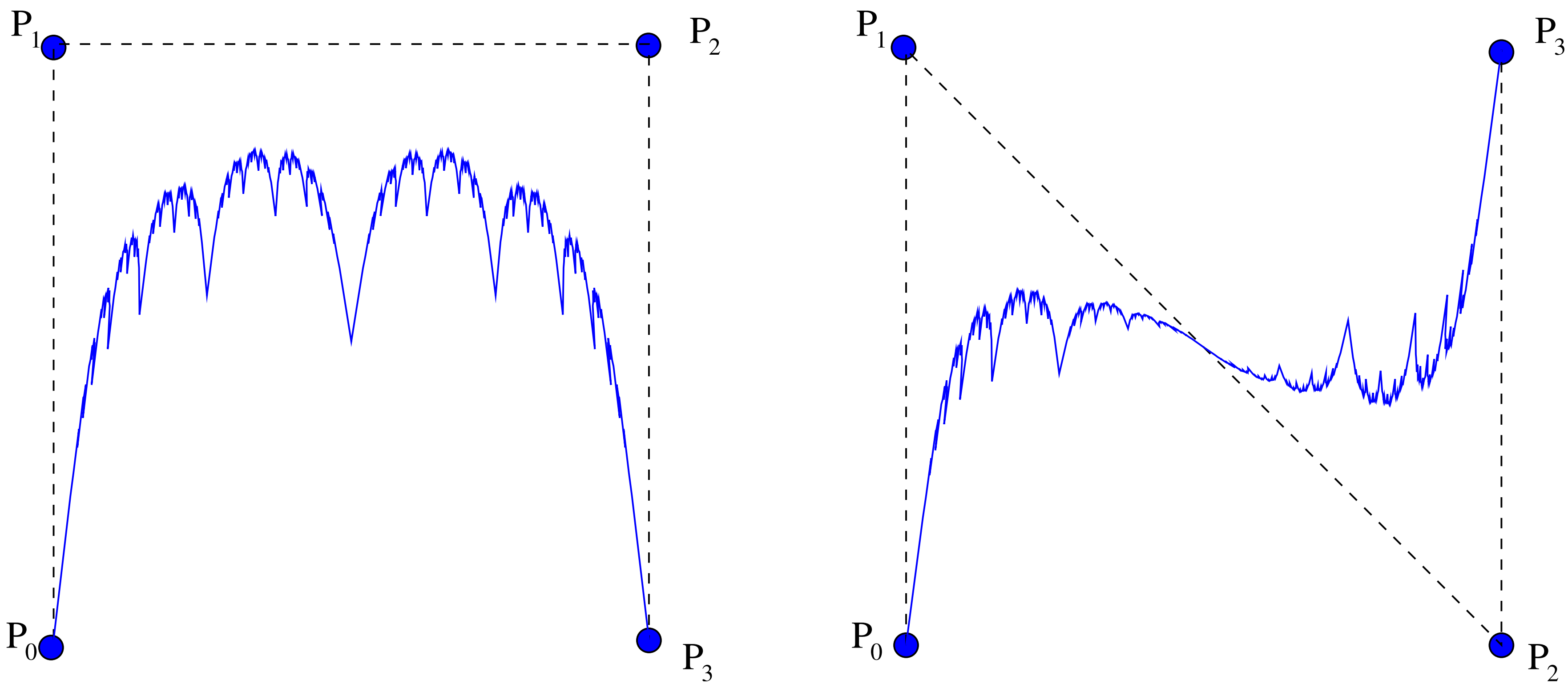
- The metric space E is a barycentric space
- T_i are barycentric columns matrices
- A set of control points $P = (p_0, \ldots, p_n)$ is used to perform a projection
- Each point of $\mathcal{A}(\mathbb{T})$ is projected:

$$P\mathcal{A}(\mathbb{T}) = P \lim_{n \to \infty} \mathbb{T}^n K$$

Projected IFS model (2)

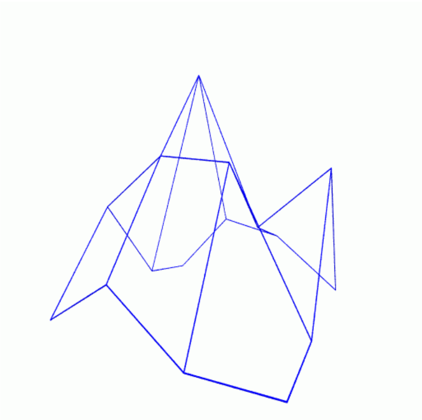
Example

- Deformation of a curve using the control points

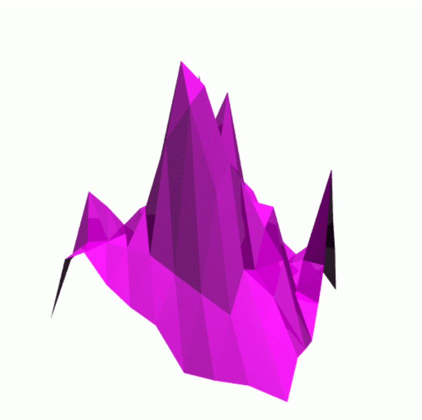


Projected IFS model (3)

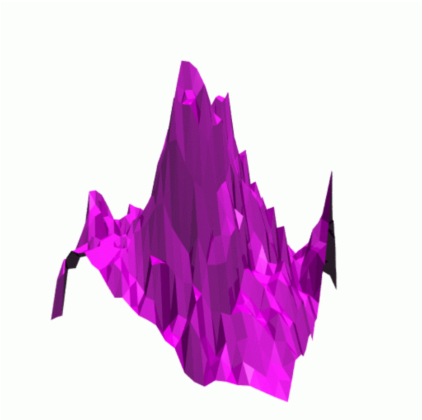
Construction algorithm



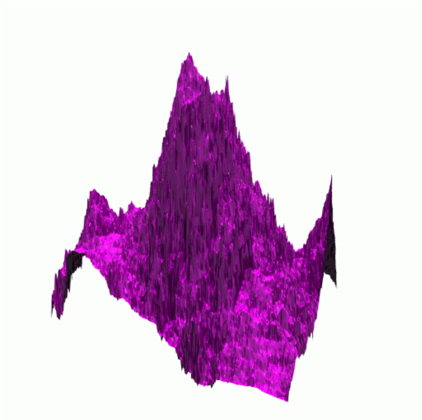
Control grid



Step 1



Step 2



Step 4

Projected IFS model (4)

Function representation

- BARNSLEY introduces an address function by indexing transformations with a finite set $\Sigma = \{0, \ldots, N-1\}^*$:

$$\sigma \in \Sigma^\omega \mapsto \phi(\sigma) = \lim_{n \to \infty} T_{\sigma_1} \ldots T_{\sigma_n} \lambda$$

- It is easy to construct a parametric representation:

$$\Phi(s) = \phi(\sigma) \text{ with } s = \sum_{i=1}^{\infty} \frac{1}{N^i} \sigma_i$$

- And the projected attractor has a function representation:

$$F(s) = P\Phi(s) = P\phi(\sigma) = P \lim_{n \to \infty} T_{\sigma_1} \ldots T_{\sigma_n} \lambda$$

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*For curves. For surfaces $\Sigma = \{0, \ldots, N-1\}^2$ is used

Approximation Method

Approximation problem formalism

- Given a sampled surface $\mathbf{Q}_{ij\,(i=0,\ldots,N^p\;j=0,\ldots,N^p)}$
- Find the projected IFS attractor that minimises:

$$\sum_{ij} ||\mathbf{Q}_{ij} - F(\frac{i}{N^p}, \frac{j}{N^p})||^2$$

- A reduced number p of iteration is needed

$$\sigma = \sigma_1 \ldots \sigma_p\, 00 \ldots$$

$$F(\frac{i}{N^p}, \frac{j}{N^p}) = PT_{\sigma_1 \tau_1} \ldots T_{\sigma_p \tau_p} e_{00}$$

Approximation Method (2)

Non-linear fitting

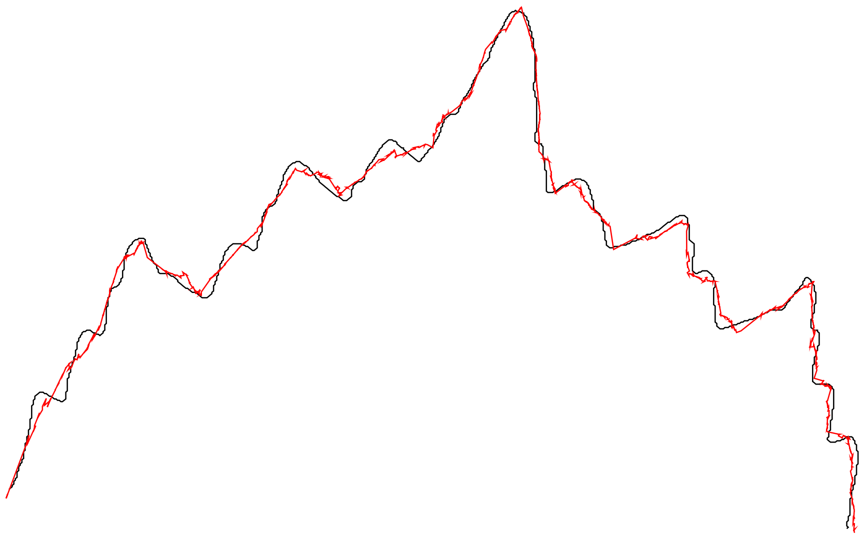
- Parameterisation of the model: parameter vector $\mathbf{a}$
  - Matrix coefficients
  - Control points coordinates
- Fractal familly function $F_{\mathbf{a}}$ issued from the couple $(P_{\mathbf{a}}, \mathbb{T}_{\mathbf{a}})$
- Approximation ➟ non-linear fitting

$$\mathbf{a}_{\text{opt}}(\mathbf{Q}) = \underset{\mathbf{a}}{\operatorname{argmin}}\, d(F_{\mathbf{a}}, \mathbf{Q})$$

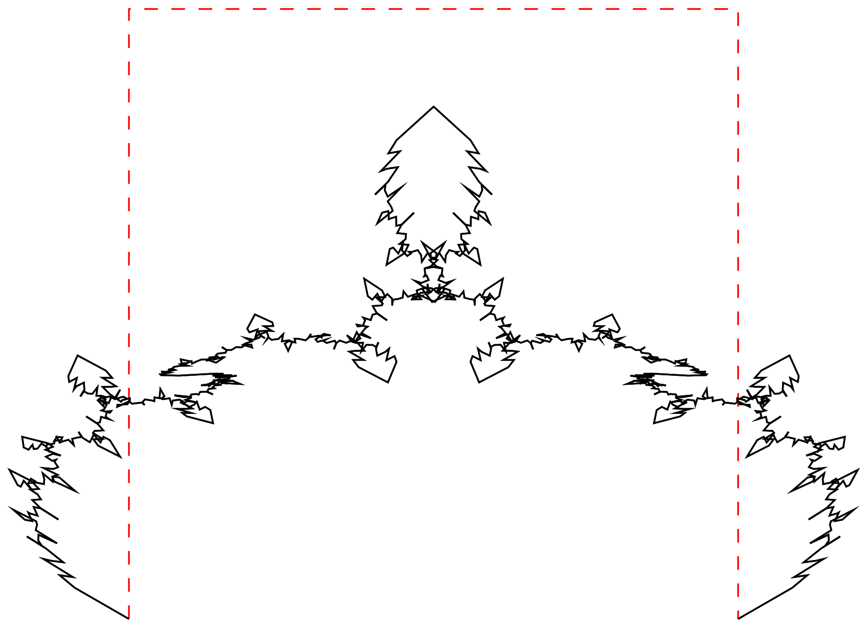
- Resolution: Levenberg-Marquardt algorithm

Results

Curve approximation



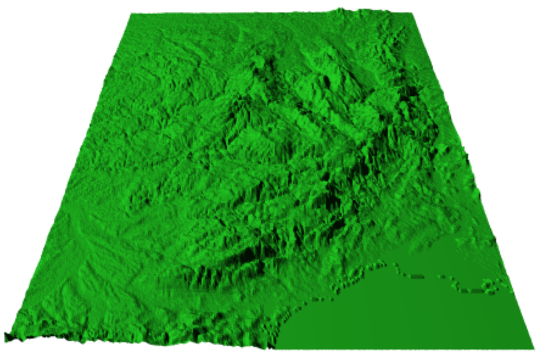
A maple leaf



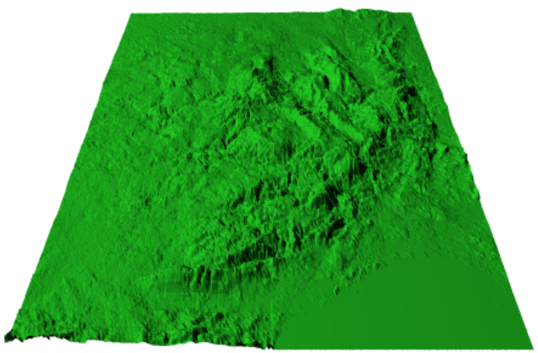
A synthetic curve

Results (2)

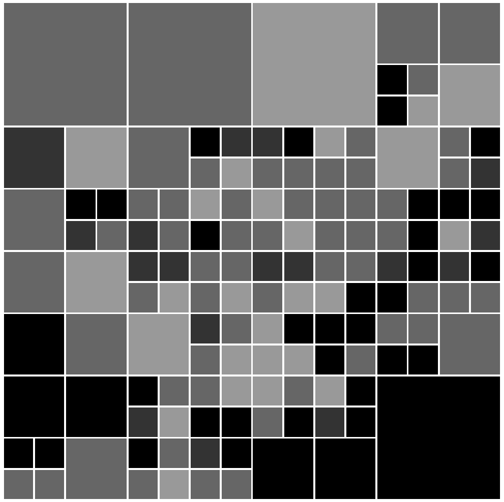
Surface approximation



Original surface



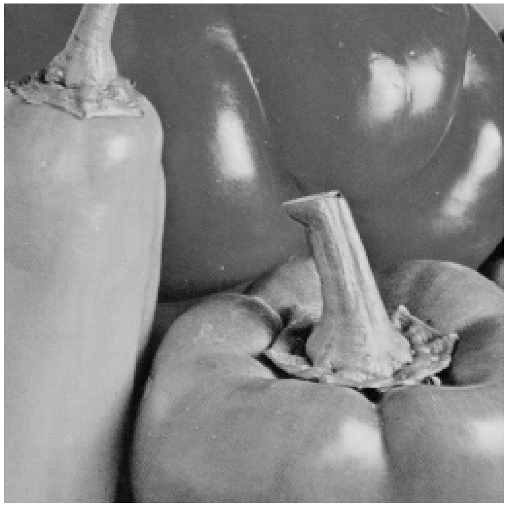
Approximated surface



Quadtree

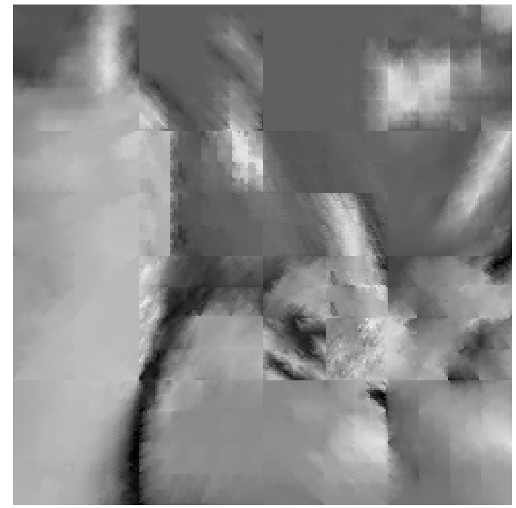
Results (3)

Image compression



Original image

JPEG2000
(0.044 bpp)
PSNR$= 23.1\, dB$



PIFS quadtree
(0.044 bpp)
PSNR$= 24.7\, dB$

Conclusion

- ✔ Flexible model: *projected IFS attractors*
- ✔ General approximation formalism through a non-linear fitting problem formulation
- ✔ Efficient approximation
- ✘ Computing time
  - ➠ A few seconds for a 100 points curve
  - ➠ More than one hour for a big image or surface (512×512 for example)
- ➠ Compromise between computing time and model complexity

Ongoing work

- Improve coding performances for greater bitrates
- Not a fractal-only coder
  - Code the error image with another method
- Introduce a function dictionnary
- Introduce hierachical modeling for curves

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Ongoing work (2)

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