Fractal Inverse Problem: Approximation Formulation and Differential Methods

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Definition and classics

- Finding a fractal code (model) that generates data (image, curve, surface, etc.)
- Based on the collage theorem [Barnsley, 1988]
- Fractal image compression [Jacquin, 1992]

A classification attempt

- Direct methods
  - Model characteristics are found directly
  - Wavelet methods [Berkner, 1997, Struzik et al., 1995]
  - Complex moment [Abiko et al., 1997]
  - Inverse problem is performed on synthetic data

A classification attempt

- Direct methods
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  - Wavelet methods [Berkner, 1997, Struzik et al., 1995]
  - Complex moment [Abiko et al., 1997]
  - Inverse problem is performed on synthetic data
- Indirect methods
  - Model characteristics are not found directly (optimization algorithm)
  - Mixed IFS and genetic algorithm [Lutton et al., 1995]

The beginning

- Differential methods
  - Derivative property of affine IFS [Vrscay and Saupe, 1999]
  - We have developed an approximation method for curves, surfaces [Guérin et al., 2000, Guérin et al., 2001], and images [Guérin et al., 2003, Guérin, 2002].

What is an address function ?

- Barnsley introduced a function that maps from infinite words Σ^ω to a modelisation space $\mathcal{X} = \mathbb{R}^m$:

$$\begin{array}{rcl} \phi : \Sigma^\omega & \rightarrow & \mathcal{X} \\ \rho & \mapsto & \phi(\rho) \end{array}$$

- Fractal context: IFS describe address functions
- Consider an indexed IFS $\mathbb{T} = (T_i)_{i \in \Sigma}$, we have the following address function:

$$\rho \in \Sigma^\omega \mapsto \phi(\rho) = \lim_{n \to \infty} T_{\rho_1} \cdots T_{\rho_n} \lambda \in \mathcal{X} \text{ with } \lambda \in \mathcal{X} \,. \quad (1)$$

Fractal approximation

- Fractal objects can be constructed with address functions
- We need a theoretical background to perform approximation on such functions
- Idea: build an Hilbert Space
- The bonus: in the case of affine IFS defined address functions, analitycity with respect to IFS affine parameters is proved
- It allows the use of non-linear fitting algorithms (gradient method, Levenberg-Marquardt,. . .)

Approximation formulation

- Approximation formulation is an optimization problem:

$$\mathbb{T}_{opt} = \underset{\mathbb{T}}{\operatorname{argmin}} \|\varphi - \psi(\mathbb{T})\|^2$$

with:

$$\|\varphi\|^2 = <\varphi, \varphi>$$

and

$$<\phi, \phi'> = \lim_{n\to\infty} \frac{1}{N^n} \sum_{\alpha\in\Sigma^n} \frac{1}{M} \sum_{k\in\Omega} <\phi(\alpha k^\omega), \phi'(\alpha k^\omega)>$$

with $M = |\Omega|$ and $N = |\Sigma|$.

$\mathcal{X} = \mathbb{R}$ and IFS are affine

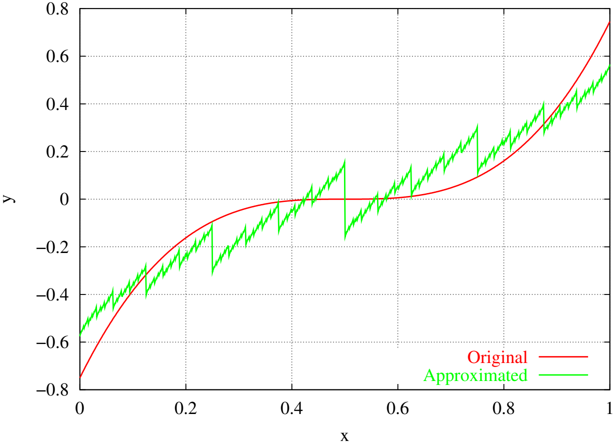
- Transformations operate on $\mathbb{R}$:

$$\begin{array}{rcl} T_i : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & a_i x + b_i \end{array}$$

- Approximation of a discrete numerical function build with pairs: $(x_i, y_i)_{i=1,\ldots,p}$
- Let $\alpha^{(i)} = \alpha_1^{(i)} \ldots \alpha_n^{(i)}$ be the N-adic expansion of $\bar{x}_i$ with $x_i = \bar{x}_i + \epsilon_i$ and $\epsilon_i < \frac{1}{N^{n+1}}$
- Approximation formulation:

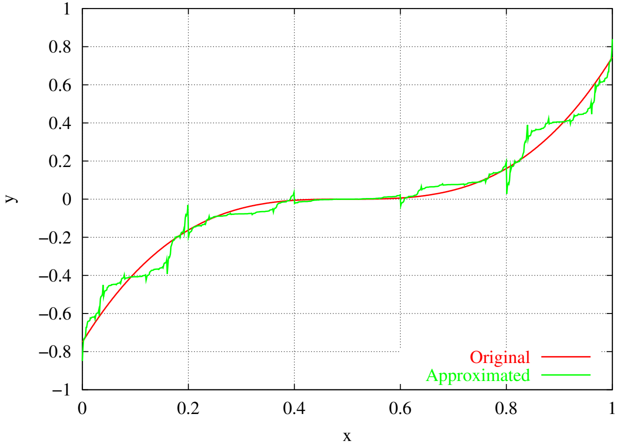
$$\mathbb{T}_{\mathrm{opt}} = \underset{\mathbb{T} \in \mathcal{S}^{\Sigma}}{\operatorname{argmin}} \frac{1}{p} \sum_{i=1 \ldots p} \left(\psi_{\alpha_1^{(i)} \ldots \alpha_n^{(i)} 0^{\omega}}(\mathbb{T}) - y_i \right)^2$$

Approximation of a cubic function (1000 points)



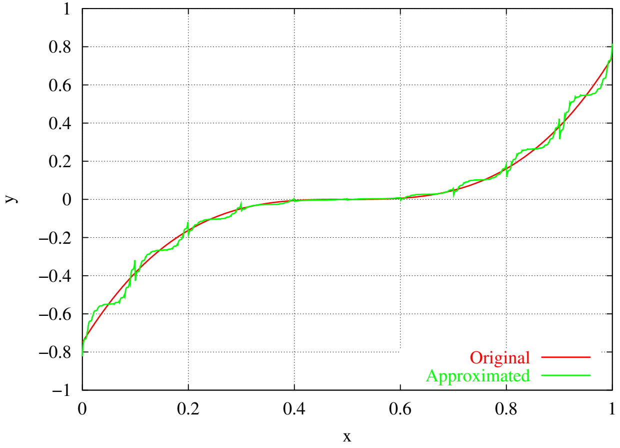
With $N = 2$

Approximation of a cubic function (1000 points)



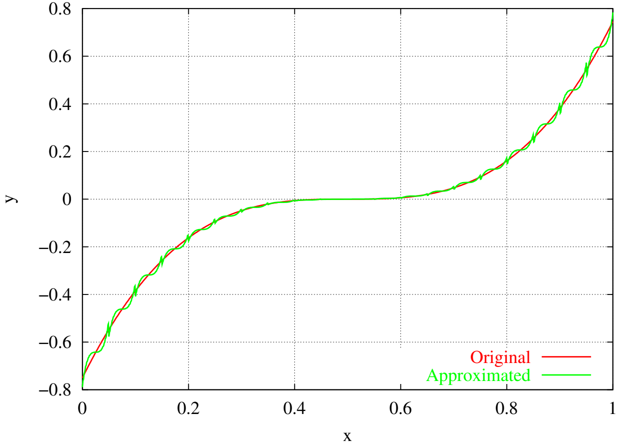
With $N = 5$

Approximation of a cubic function (1000 points)



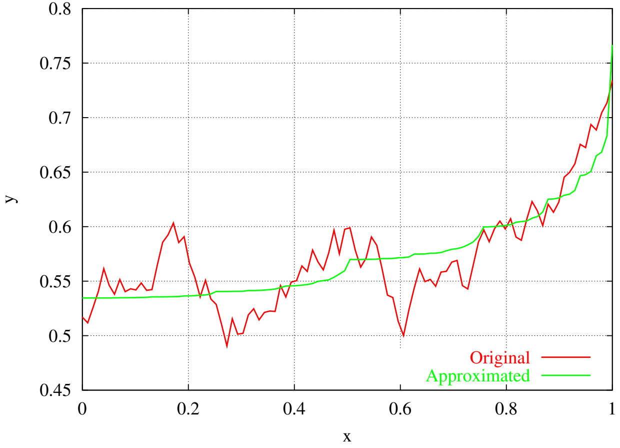
With $N = 10$

Approximation of a cubic function (1000 points)



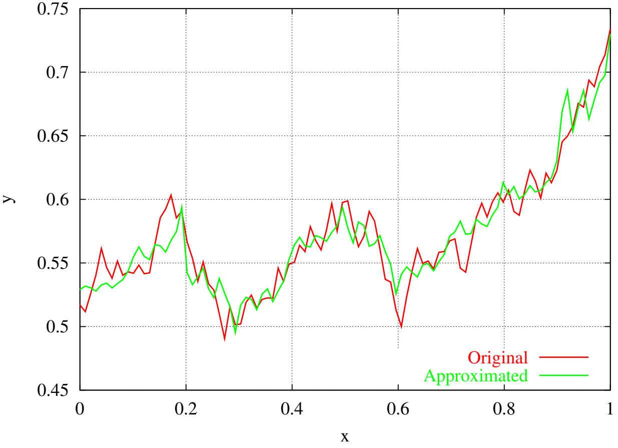
With $N = 20$

Approximation of a random function (100 points)



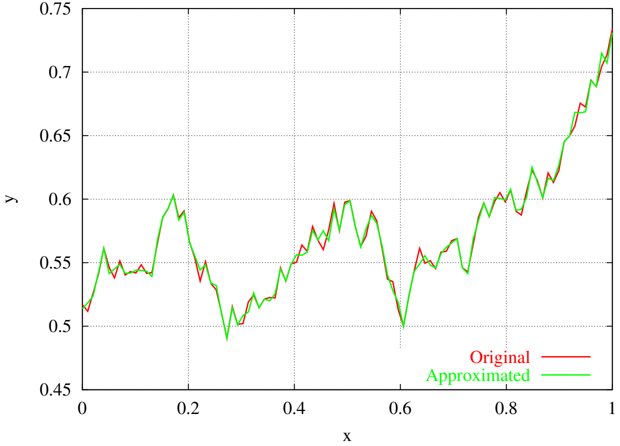
With $N = 2$

Approximation of a random function (100 points)



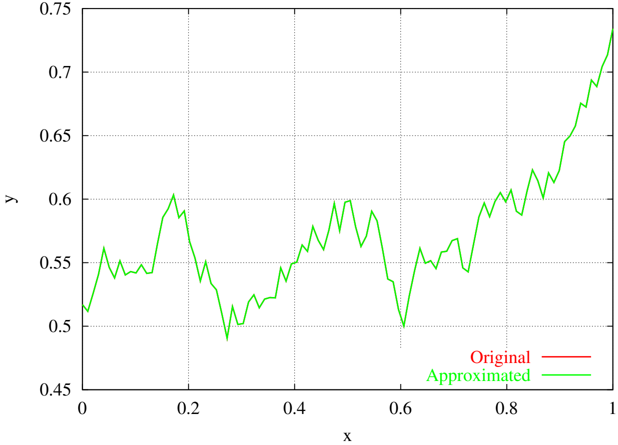
With $N = 10$

Approximation of a random function (100 points)



With $N = 40$

Approximation of a random function (100 points)



With $N = 100$

Projected IFS model

- Iteration space $\mathcal{X}$ is barycentric:

$$\mathcal{X} = \mathcal{B}^J = \{(\lambda_j)_{j \in J} \mid \sum_{j \in J} \lambda_j = 1\}$$

- Iteration semigroup is constituted of matrices with barycentric columns:

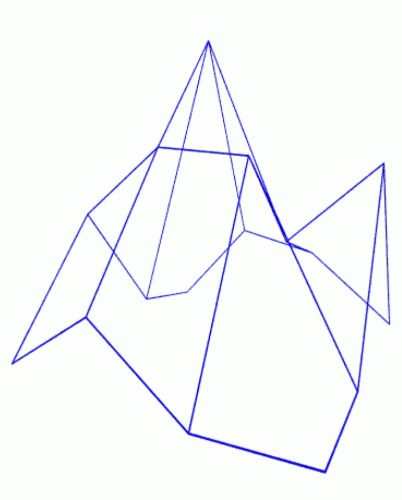
$$S_J = \{T \mid \sum_{j \in J} T_{ij} = 1, \forall i \in J\}$$

- Attractor is projected through control points:

$$P\mathcal{A}(\mathbb{T}) = \{P\lambda \mid \lambda \in \mathcal{A}(\mathbb{T})\}$$

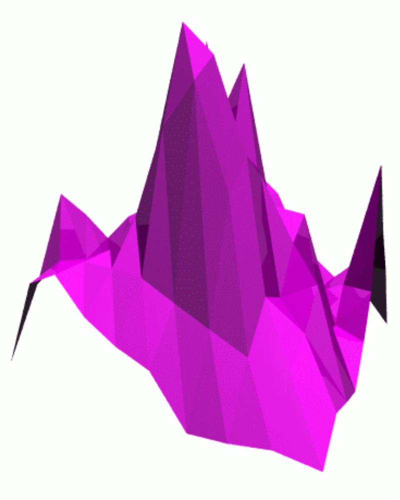
where P is a control polygon or grid.

Construction algorithm



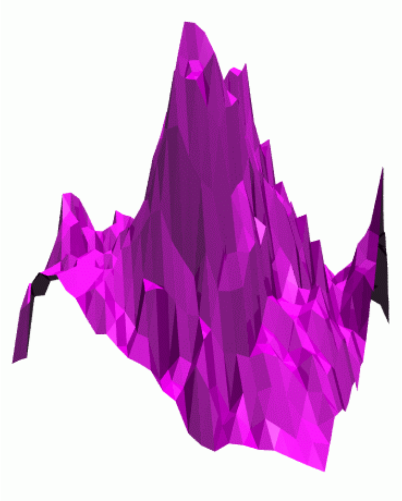
Control grid

Construction algorithm



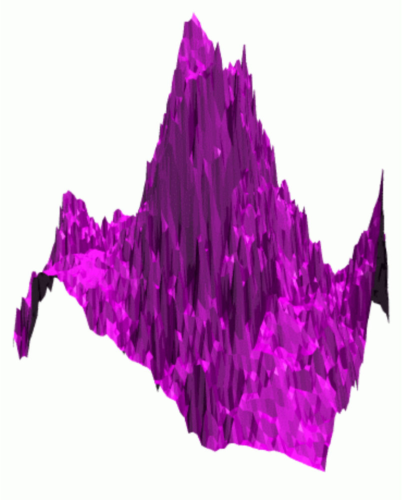
Step 1

Construction algorithm



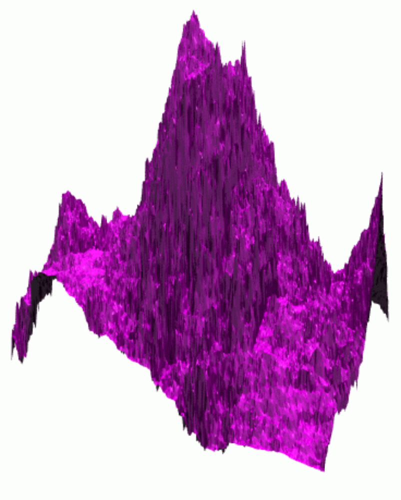
Step 2

Construction algorithm



Step 3

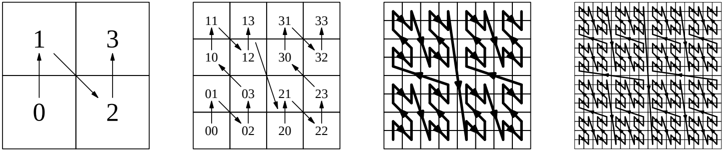
Construction algorithm



Step 4

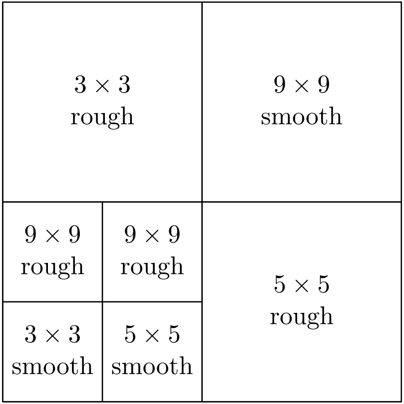
Address function

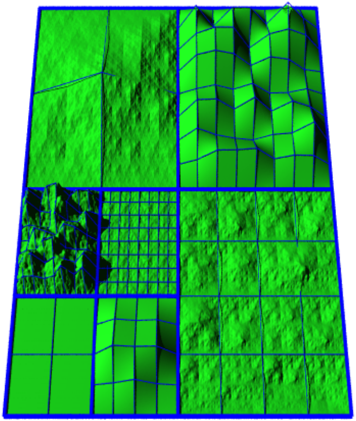
- Easy to provide with a Péano mapping:



Extension : Projected IFS tree model

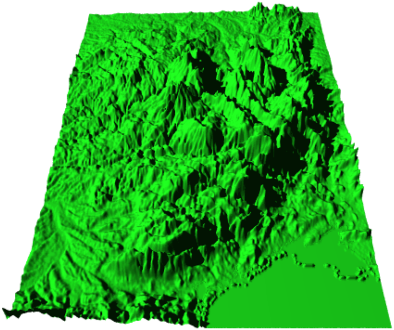
- Each leaf γ of the quadtree contains a complete projected IFS model pair $(\mathbb{T}^\gamma, P^\gamma)$
- Allows the combination of local smooth/rough behavior


3×3
rough
9×9
smooth
9×9
rough
9×9
rough
5×5
rough
3×3
smooth
5×5
smooth

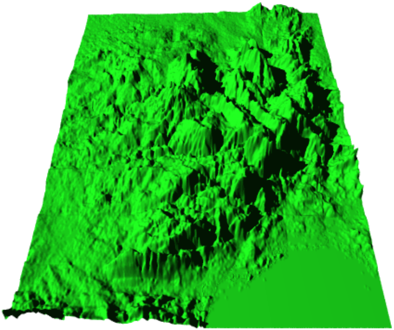


Surface approximation

- Each leaf of the quadtree is subdivided until an error threshold is reached
- Example: the threshold is fixed at $PSNR > 40dB$



Original



Approximation

Image compression

- A recursive and exhaustive coding process is performed with several model type and several quantification levels
- A final optimization step is performed with a bit-rate goal (bisection algorithm)
- Example: Image compressed at 0.12bpp, PSNR=28.3dB

Original

Compressed

Conclusion

- Address functions constitute a general framework for approximation
- Affine IFS coefficients map to attractors with analytical functions
- Standard non-linear optimization algorithms can be employed
- IFS extensions allow easy curve, surface and image modeling
  - Projected IFS
  - Projected IFS trees
- Approximation results have been obtained
  - Curves
  - Surfaces
  - Image (compression)

Ongoing work

- Combine non-linear optimization and wavelet approach for better surface representation and compression
- Propose a common formalism for subdivision surfaces (from CAGD) and IFS
- Extend the possibility to treat surfaces with complex topology
- ...

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