FRACTAL APPROXIMATION AND COMPRESSION USING PROJECTED IFS

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Abstract: *Approximation of natural objects (curves, surfaces, or images) with fractal models is an important center of interest for research. The general inverse problem paradigm concerns many application fields and a large variety of studies have been proposed to address it.The most known of them is the fractal image compression method introduced by Jacquin. Generally speaking, these techniques lack of flexibility in term of control over the approximated shape. Furthermore, iteration space used is the visualisation space,* $\mathbb{R}^2$*. Previous work achieved a general framework for fractal modeling: fractal free forms. This model allows user to define self-similar objects in a space of a higher dimension. We propose a resolution of the inverse problem based on this model and a non-linear regression algorithm. A hierachical extension of this model is introduced for modeling heterogeneous objects, for which characteristics are varying in space. A complete coding scheme has been performed on such a model showing good performances for low bitrate compression.*

Key words: *fractal, projected IFS, approximation, image compression, quadtree, non-linear fitting*

1. INTRODUCTION

Models that are able to produce rough objects (curves, surfaces, images,...) are mostly based on random processes. This is the reason why these models are not suitable for approximation. In order to propose an efficient solution to the problem of rough objects approximation, the current study proposes a parametric model based on a deterministic fractal approach.

In [11] and [12], we have proposed a model for fractal curves and surfaces. This model combines two classical models: a fractal model (Iterated Function Systems attractors) and a CAGD model (free form shapes). This model is called projected IFS model. A set of control points allows an easy and flexible control of the fractal shape generated by the IFS model and provide a high quality fitting, even for surfaces with sharp transitions. In [2] and [3], we have proposed an approximation method for curves based on this model. In [4] and [5], we give the extension of this method to surfaces. In this paper, we present a general framework for surface and image modeling, using a combination of projected IFS attractors with quadtrees.

2. PROJECTED IFS MODEL

In this section we develop the projected IFS attractors model. First, we introduce the IFS model. Then, we see how it is possible to obtain a parametric representation of this model. Afterward, we show how this IFS attractor can be projected through control points. At least, tabulation of the surface is introduced to simplify computations.

2.1. IFS

Introduced by BARNSLEY[1] in 1988, the IFS (Iterated Function Systems) model generates a geometrical shape or an image [6] with an iterative process. An IFS-based modeling system is defined by a triple $(\mathcal{X}, d, \mathcal{S})$ where:

- $(\mathcal{X}, d)$ is a complete metric space, $\mathcal{X}$ is called *iteration space*;
- $\mathcal{S}$ is a semigroup acting on points of $\mathcal{X}$ such that: $\lambda \in \mathcal{X} \mapsto T\lambda \in \mathcal{X}$ where T is a contractive operator, $\mathcal{S}$ is called *iteration semigroup*.

An *IFS* $\mathbb{T}$ *(Iterative Function System)* is a finite subset of $\mathcal{S}$: $\mathbb{T} = \{T_0, ..., T_{N-1}\}$ with operators $T_i \in \mathcal{S}$. We note $\mathcal{H}(\mathcal{X})$ the set of non-empty compacts of $\mathcal{X}$. The associated HUTCHINSON operator is:

$$K \in \mathcal{H}(\mathcal{X}) \quad \mapsto \quad \mathbb{T}K = T_0 K \cup ... \cup T_{N-1} K \quad .$$

This operator is contractive in the new complete metric space $\mathcal{H}(\mathcal{X})$ and admits a fixed point, called *attractor* [1]:

$$\mathcal{A}(\mathbb{T}) = \lim_{n\to\infty} \mathbb{T}^n K \; with \; K \in \mathcal{H}(\mathcal{X})\,.$$

2.2. Parameterisation of attractors

By introducing a finite set Σ, the IFS can be indexed $\mathrm{T} = (T_i)_{i\in\Sigma}$ and the attractor $\mathcal{A}(\mathbb{T})$ has an *address function* [1] defined on Σ^ω, the set of infinite words of Σ:

$$\sigma \in \Sigma^\omega \mapsto \phi(\sigma) = \lim_{n\to\infty} T_{\sigma_1}...T_{\sigma_n}\lambda \in \mathcal{X} \text{ with } \lambda \in \mathcal{X}\,. \tag{1}$$

When operators match joining condition [11, 12], this function defines parameterised curves or surfaces. For curves, a single indexing $\tilde{\Sigma} = \{0, ..., N-1\}$ is sufficient [8, 7]:

$$\Phi(s) = \phi(\sigma) \text{ with } s = \sum_{i=1}^{\infty} \frac{1}{N^i}\sigma_i$$

where $\sigma = \sigma_1 \dots \sigma_n \dots$ corresponds to the development of the scalar s in base N. For surfaces, it is more convenient to use PEANO indexing $\Sigma = \{0, \dots, N^2-1\}$:

$$\Phi(s,t) = \phi(\rho) \;\text{ with }\; \rho = \sigma_1\natural\tau_1 \dots \sigma_n\natural\tau_n \dots \in \Sigma^\omega$$

where $\sigma = \sigma_1 \dots \sigma_n \dots$ and $\tau = \tau_1 \dots \tau_n \dots$ are respectively the development of s and t in base N and $\sigma_i\natural\tau_i = N\sigma_i + \tau_i$.

2.3. Projected attractors

The main idea of our model is drawn from the formula of free form surfaces used in CAGD:

$$F(s,t) = \sum_{j\in J} \Phi_j(s,t) p_j$$

where p_j constitutes a grid of control points (see Fig. 1), and Φ_j are blending functions. These blending functions have the following property:

$$\forall (s,t) \in [0,1]^2 \quad \sum_{j\in J} \Phi_j(s,t) = 1\,.$$

The way to obtain the same property for IFS attractors is to use a barycentric metric space $\mathcal{X} = \mathcal{B}^J$[11, 12]:

$$\mathcal{B}^J = \{(\lambda_j)_{j\in J} \,|\, \sum_{j\in J} \lambda_j = 1\}\,.$$

For curves, this barycentric space is used with $\tilde{J} = \{0, \dots, m\}$, for surfaces with $J = \{0, \dots, m\} \times \{0, \dots, m\}$. Then, the iteration semigroup is constituted of matrices with barycentric columns:

$$S_J = \{T \,|\, \sum_{j\in J} T_{ij} = 1, \, \forall i \in J\}\,.$$

This choice leads to the generalization of IFS attractors named *projected IFS attractors*:

$$P\mathcal{A}(\mathbb{T}) = \{P\lambda \,|\, \lambda \in \mathcal{A}(\mathbb{T})\}$$

where P is a grid of control points $P = (p_j)_{j\in J}$ and $P\lambda = \sum_{j\in J} \lambda_j p_j$. In this way, we can construct a fractal function [11, 12] using the projection:

$$F(s,t) = P\Phi(s,t) = \sum_{j\in J} \Phi_j(s,t) p_j$$

where $\Phi(s,t)$ is a vector of functions $\Phi(s,t) = (\Phi_j(s,t))_{j\in J}$ and J is the double index set $J = \{0, \dots, m\} \times \{0, \dots, m\}$. Two different actions may be performed on the model, each one playing a different role (see Fig. 1):

- Action on the control grid P performs a global deformation on the surface.
- Action on the transformations $\mathbb{T}$ changes the local aspect of the surface (roughness).

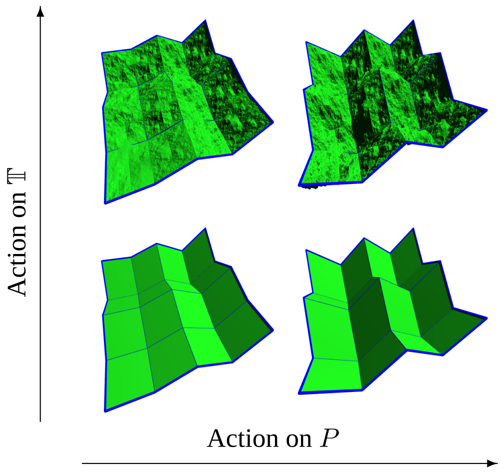


Fig. 1 Deformation of a free form surface using the control grid.

2.4. Tabulation of parametric surfaces

With a tabulation process[3, 4], considering only the values of s and t multiple of $\frac{1}{N^p}$ leads to a simplification in the computing without any loss of information. The surface tabulation is a grid defined by:

$$F(\frac{i}{N^p}, \frac{j}{N^p}) = P\Phi(\frac{i}{N^p}, \frac{j}{N^p}) \text{ with } (i,j) \in \{0, \ldots, N^p - 1\} \times \{0, \ldots, N^p - 1\}\,.$$

In this special case, developments of $\frac{i}{N^p}$ are ended by a infinite sequences of 0:

$$\left\{ \begin{array}{lcl} \sigma & = & \sigma_1 \ldots \sigma_p\, 00 \ldots \\ \tau & = & \tau_1 \ldots \tau_p\, 00 \ldots \end{array} \right.$$

Denoting $\rho_i = \sigma_i \natural \tau_i$ simplifies $F(\frac{i}{N^p}, \frac{j}{N^p})$ in:

$$\begin{array}{rcl} F(\frac{i}{N^p}, \frac{j}{N^p}) & = & P\phi\,((\sigma_1 \natural \tau_1) \ldots (\sigma_p \natural \tau_p) \ldots (0 \natural 0) \ldots (0 \natural 0) \ldots) \\ & = & PT_{\rho_1} \ldots T_{\rho_p} \phi\,(0 \ldots 0 \ldots) = PT_{\rho_1} \ldots T_{\rho_1} \Phi(0,0) \end{array}$$

By choosing simplifications (but no restrictions) such as $\Phi(0,0) = e_{00}$, the surface tabulation can be generated computing only p iterations without any loss of information:

$$F(\frac{i}{N^p}, \frac{j}{N^p}) = PT_{\rho_1} \ldots T_{\rho_p} e_{00}\,.$$

Fig. 2 shows the three first iterations of the construction process.

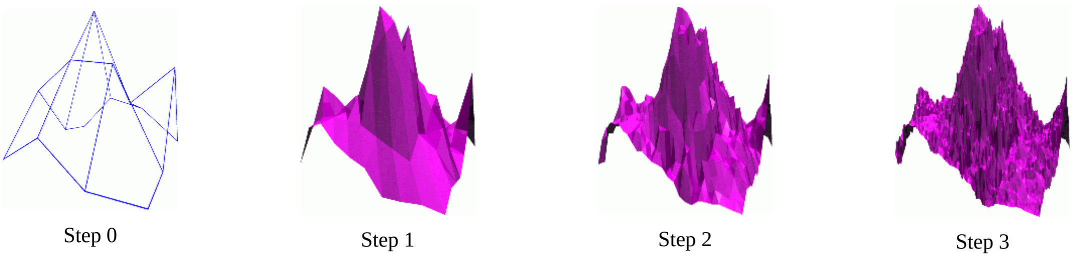


Fig. 2 Three first iterations of the construction process

3. QUADTREE OF PROJECTED IFS MODELS

The previous section shows that the use of Projected IFS attractors implies a uniform control on the form. This is a real restriction when the goal is approximation. In this section, we introduce a generalization of projected IFS attractors by adding a quadtree structure.

Let us denote Γ a cut of the quadtree and $\gamma \in \Gamma$ a leaf node of this cut. As the indexing of this leaf node is similar to the indexing of transformations of IFS, we will use the same notation, *i.e.* $\gamma \in \Sigma^*$ with $\Sigma = \{0, 1, 2, 3\}$. We introduce subdivision functions of $I = [0, 1]$:

$$\mathrm{T}_0^I(s) = \frac{1}{2}s \text{ and } \mathrm{T}_1^I(s) = \frac{1}{2}(s+1),\ s \in I$$

Then, it is possible to introduce subdivision functions of $[0, 1]^2$:

$$\begin{array}{lcl} \mathrm{T}_0(s,t) = (\mathrm{T}_0^I(s), \mathrm{T}_0^I(t)) & ; & \mathrm{T}_1(s,t) = (\mathrm{T}_0^I(s), \mathrm{T}_1^I(t)) \\ \mathrm{T}_2(s,t) = (\mathrm{T}_1^I(s), \mathrm{T}_0^I(t)) & ; & \mathrm{T}_3(s,t) = (\mathrm{T}_1^I(s), \mathrm{T}_1^I(t)) \end{array}$$

with $(s, t) \in [0, 1]^2$.

We associate to each γ a projected IFS model: control grid P^γ and IFS $\mathbb{T}^\gamma = (T_i^\gamma)_{i\in\Sigma}$. The fractal surface is then defined by projected IFS patches organized in a quadtree structure:

$$F(s,t) = \sum_{\gamma\in\Gamma} \chi_\gamma(s,t)\ P^\gamma\ \Phi^\gamma(\mathrm{T}_\gamma^{-1}(s,t))$$

where

$$\chi_\gamma(s,t) = \begin{cases} 1 & \text{if } (s,t) \in \mathrm{T}_\gamma[0,1]^2 \\ 0 & \text{else} \end{cases}$$

and $\mathrm{T}_\gamma = \mathrm{T}_{\gamma_1} \circ \ldots \circ \mathrm{T}_{\gamma_k}$. To obtain a continuous function, a boundary constraint has to be satisfied. We will not detail this joining condition in this paper.

4. APPROXIMATION METHOD

First we shows how it is possible to perform an approximation on a single model by a non-linear fitting formalism. Assuming we are able to approximate a given surface with a given projected IFS model, our goal is then to provide an adaptative method for approximating complex surfaces and images with a quadtree of projected IFS models.

4.1. Projected IFS model approximation

Given a sampled surface $(s_i, t_j, \mathbf{Q}_{ij}) \in \mathbb{R}^3$, the challenge is to determine the projected IFS model which provides a good quality approximation of this surface. The approach proposed in the current study is similar to the one we introduced in [2, 3] for curves. It is based on a non-linear fitting formalism.

Let $\mathbf{Q}_{ij\,(i=0,\ldots,N^p\ j=0,\ldots,N^p)}$ be a given surface to approximate. Let $F_\mathbf{a}$ be the function associated with the parameter vector $\mathbf{a}$. The approximation problem consists in determining the parameter vector $\mathbf{a}$ that minimizes the distance between the sampled surface $\mathbf{Q} = \{(\frac{i}{N^p}, \frac{j}{N^p}, \mathbf{Q}_{ij})\}$ and the function $F_\mathbf{a}$:

$$\mathbf{a}_{opt} = \underset{\mathbf{a}}{\operatorname{argmin}}\ d(\mathbf{Q}, F_\mathbf{a})$$

where:

$$d(\mathbf{Q}, F_\mathbf{a}) = \sum_{ij} ||\mathbf{Q}_{ij} - F_\mathbf{a}(\frac{i}{N^p}, \frac{j}{N^p})||^2 .$$

Our resolution method is based on the LEVENBERG-MARQUARDT algorithm [9]. This algorithm is a numerical resolution of the following fitting problem:

$$\mathbf{a}_{\text{opt}} = \underset{\mathbf{a}}{\operatorname{argmin}} \sum_{i=0}^{M} (v_i - f(\mathbf{a}, u_i))^2$$

where vectors $\mathbf{v}$ and $\mathbf{u}$ are the fitting data and f is the fitting model.

(a) Original Image

(b) JPEG2000, $0.044bpp$, PSNR $= 23.1dB$

(c) PIFS, $0.044bpp$, PSNR $= 24.7dB$

Fig. 3 Compression results

In order to resolve our approximation problem using this algorithm, we have to consider the following data:

$$\mathbf{v} = (v_0, \ldots, v_M) = (0, \ldots, 0) \ ; \ \mathbf{u} = (u_0, \ldots, u_M) = (0, \ldots, M)$$

where $M = (N^p + 1)^2$. Then, the fitting model is:

$$f(\mathbf{a}, k) = ||\mathbf{Q}_{i_k j_k} - F_{\mathbf{a}}(\frac{i_k}{N^p}, \frac{j_k}{N^p})||$$

where $i_k = k \mod N^p$ and $j_k = k/N^p \ \ \forall k = 0, \ldots, (N^p + 1)^2$.

The LEVENBERG-MARQUARDT method combines two types of approximation (linear and quadratic) for minimizing the square distance. These approximations are computed with the provided partial derivatives of the fitting model. In our case, these partial derivatives are numerically computed by a perturbation vector [2, 3]:

$$\frac{\partial f}{\partial a_i}(\mathbf{a}, u) \simeq \frac{f(\mathbf{a} + \delta\mathbf{a}_i, u) - f(\mathbf{a}, u)}{\varepsilon}$$

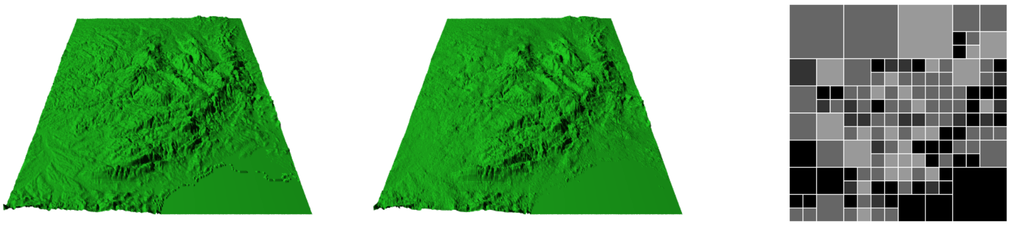
with $\delta\mathbf{a}_i = (\underbrace{0, \ldots, 0, \varepsilon}_{i}, 0, \ldots, 0)$

4.2. Quadtree approximation

We are able to perform approximation on a piece of surface or image. The final goal is to find the quadtree structure that approximate a whole surface or image, given a criteria. For grey-level images, the standard criteria is Rate/Distortion ratio optimisation. That means, given a coding bitrate (quantity of information needed for coding), find the description that minimizes the distortion (error) between the original and the reconstructed images. We have implemented an optimisation algorithm based on [10] with a Lagrange multiplier formalism. For surfaces, it is more convenient to give a distortion criteria, and to find the simplest model that satisfies this constraint. Results of a surface approximation and an image compression are shown in the next section.

5. RESULTS

Fig. 3 shows an example of image compression. Original image (Fig. 3a) has been compressed with two methods: the standard JPEG 2000 (Fig. 3b) and our method (Fig. 3c). The coding bitrate is $0.044bpp$, it represents a compression ratio of $1 : 181$. At this very low bitrate, our method generates less artifacts and a smaller distortion. Fig. 4 shows a surface approximation result. The original surface (Fig. 4a) is the french "Massif Central". Fig. 4b shows its approximation with a distortion criteria of $35\,dB$. Fig. 4c shows the quadtree structure generated for this example. Dark patches represent simple models since we use several types of model.



(a) Original surface, french "Massif Central"

(b) Approximated surface

(c) Quadtree

Fig. 4 Surface approximation results

6. CONCLUSION

We presented a new approach for modeling both rough or smooth objects. This method is based on a fractal model named projected IFS attractors. This model is a parametric description which has the advantage of compactly describing the surface shape making it useful for geometric modeling and image synthesis. Several projected IFS models are combined in a quadtree structure to obtain more accurate and adaptative modeling. Results show that our method is an interesting approach for low bitrate image compression and surface approximation.

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