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Mathematical framework for topological relationships between ribbons and regions ☆

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ABSTRACT

In a map, there are different relationships between spatial objects, such as topological, projective, distance, etc. Regarding topological relations, if the scale of the map is changed and if some spatial objects are generalized, not only the shapes of those objects will change (for instance, a small area becomes a point and then disappears as the scale diminishes), but also their topological relations can vary according to scale. In addition, a mathematical framework which models the variety of this category of relationships does not exist. In the first part of this paper, a new topological model is presented based on ribbons which are defined through a transformation of a longish rectangle; so, a narrow ribbon will mutate to a line and then will disappear. Suppose a road is running along a lake, at some scales, they both appear disjointed whereas at some smaller scales, they meet. So, the topological relations mutate according to scale. In this paper, the different components of this mathematical framework are discussed. For each situation, some assertions are defined which formulate the mutation of the topological relationships into other ones when downscaling.

1. Introduction

When somebody is saying “this road runs along the sea,” what are exactly the spatial or geographical relations which are concerned? Sometimes, either the road touches the sea or a small beach is located between the road and the sea, etc. From a mathematical point of view, mostly there is a disjointed relation between the road and the sea whereas for people the relation is different. In addition, when one is reading a map, according to scale, the topological relation can be different, disjoint or meet. So, topological relations can vary according to scale. Suppose a decision-maker wants to create a new motorway running along a lake with the help of a computer. Taking this consideration into account, any reasoning system will generate difficulties because the spatial relations hold differently: any conceptual framework dealing with spatial relationships must be robust against scales.

Another problem comes from mathematical modeling of streets and rivers. Often, they are considered as linear objects even if they have some widths or areas. By considering a road as a line or as an area, topological relationships can be different. In order to solve this problem, the concept of ribbon will be developed. Depending on the scale, or more exactly on visual acuity and granularity of interest, a ribbon will be a longish rectangle (area), a line or it will disappear. In other words, ribbons can be seen as an extension of polylines. Moreover, in order not to be stuck to cartography, the concept of granularity of interest will be introduced.

This paper will be organized as follows. Firstly, definitions and a state of the art review for the generalization

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process will be given (Section 2). Also, definitions and a state of the art review for topological relationships will be given (Section 3). Then, the framework of topological relationships for ribbons will be defined (Section 4). Finally, we present a conclusion and future work (Section 5).

2. Geographic object generalization

2.1. Definition

Many definitions have been given for the generalization process, The International Cartographic Association [12] has defined it as "the selection and simplified representation of detail appropriate to scale and/or the purpose of a map." The geographic object generalization is a very complex process. In order to reduce its complexity, the overall process is often decomposed into individual subprocesses, called operator [13], such as simplification, displacement etc. Each operator defines a transformation that can be applied to a single spatial object, or to a group of spatial objects.

2.2. State of the art for generalization process

Historically, cartographic production was a matter of cartographers who generate maps for different users, generally for a specific domain (e.g., geological maps). This cartographic production always includes steps of possible control to ensure the quality of the generalized map. This crucial phase is usually performed by experienced cartographers. But today, with computer technologies that allow users not experts in the field of cartographic production to generate maps without the intervention of experts.

The first generalization process appeared in early 1990s [14]. It involved only a few geographical areas. The first algorithm for generalizing polylines was published in Ref. [15]. Then, several variants were published essentially to improve the results of the initial algorithm. However, this algorithm does not take into account many aspects, such as the topological relationships between objects.

Now, several methods and concepts have been proposed to model and implement the generalization process but a framework for their combination into a comprehensive generalization process is still missing [16].

Ruas and Plazanet [17] proposed a framework controlled by a set of constraints. The dynamic generalization model is based on avoiding constraint violations and on the local qualification of a set of objects represented by means of an object situation. A situation is described by the geographical objects involved, their relationships and the constraint violations. They concentrated only on constraints related to objects and not on the constraints between objects such as the topological constraints.

Many other works use the least squares adjustment theory to solve the generalization problems such as [18,19,20]; these works aim to globally reduce all spatial conflicts. The idea is to solve spatial conflicts by modeling different constraints using mathematical expressions. Moreover, Harrie [21] proposed to formulate the geometrical and topological constraints as linear functions of the object coordinates. The least squares adjustment seems to be an interesting technique but these constraints are difficult to express by a linear equation.

In the same context and for reducing the spatial conflicts in the map, many interesting methods were proposed in [21,22]. In those approaches, a cost function (fitness) must be defined for validating the statements. However, it is questionable whether it is realistic to define such a function that integrates all the constraints of generalization such as the topological constraints.

Then several works model the spatial objects by agents such as the works of [23–25]. In the agent-based model, the spatial objects are modeled by the decisional entities in the generalization system. These entities are software agents the goal of which is to satisfy their cartographic constraints as much as possible. In Ruas [23], the constraints are subdivided into four types: metric, topological, structural and procedural constraints. The topological constraints ensure that any topological relationship between objects is maintained or modified consistently, for example, self-intersections of an object or any intersection between two objects must be avoided.

Also to improve the map generalization process, another approach was proposed in [26], which is based on a new concept called SGO (self-generalizing object). An SGO is able to generalize a cartographic object automatically using one or more geometrical patterns, simple generalization algorithms and spatial integrity constraints, but this approach does not define a pattern for topological constraints.

In the EuroSDR project, cartographic experts of four NMAs (National Mapping Agencies) were called to evaluate the results of the automation generalization process according to certain constraints [29]. The objective of this project is to illustrate the state-of-the-art of automated generalization in practice, exchange of knowledge between research community, NMAs and software vendor and to contribute to the development of constraint specification. Four test cases were selected and provided by the participating NMAs. The NMAs defined their map specifications for automated generalization in template which were developed by the EuroSDR team [29]. These map specifications were formalized as a set of cartographic constraints to be followed. They distinguished between two main categories of constraints: legibility constraints and preservation constraints. After the analysis of constraints composition, the EuroSDR project team derived a list of generic and specific cartographic constraints which must be respected in the generalization process.

Lejdel and Kazar [27] proposed an approach for optimizing the automatic generalization process by satisfying cartographic constraints. This approach consists of providing agents with geographical genetic properties to enable them to choose the optimal actions, thus giving the concept of genetic agent. Each geographical agent is equipped with an optimizer, and each one executes a genetic algorithm to determine the optimal action to be executed according to its current state in order to satisfy cartographic constraints as much as possible. The genetic algorithm follows the classical steps as selection, crossover and mutation. The solution is refined gradually over the iterations until reaching convergence to a solution that approaches the optimal solution and a certain degree of imperfection is acceptable. The solution here is a set of algorithms with adapted parameters which minimize conflicts. The model of the topological constraints of this approach is not addressed in this paper.

3. Topological relations

3.1. Definition

Topology is defined as the mathematical study of the properties that are preserved through deformations of objects. Many works can be cited here such as the work of Thom and Zeeman [30]; they study the evolution of forms in nature. Thus, this theory can be applied in mapping and more exactly in the transformation of topology.

Topology is foremost a branch of mathematics, but some concepts are of importance in the GIS domain, such as topological relationships [1]. Topological relationships describe relationships between all objects in space, the points, lines and areas for all possible kinds of deformation. Several researchers have defined topological relationships in the context of geographical information [2–4].

3.2. State of the art for topological relations

From a historical point of view, different topological models were proposed. Firstly, Allen [5] proposed a model organizing pieces of a linear model which can also be used for temporal reasoning, Max Egenhofer [6] with his colleagues proposed the first topological model for two-dimensional objects, and then Lee and Hsu [7,8] defined the relations between rectangles. Let us examine them rapidly.

3.2.1. Allen model

The objective of the Allen model is to represent the relations between two segments [5], as illustrated in Fig. 1.

3.2.2. Egenhofer region topological relationship

To define a model of topological relationships, Egenhofer and Herring [6] proposed a spatial data model based on topological algebra. The algebra topological model is based on geometrical primitives called cells that are defined for different spatial dimensions 0-D, 1-D, and 2-D. A variety of topological properties between two cells can be expressed in terms of the 9-intersection model [10]. The 9-intersection model between two cells A and B is based on the combination of six topological primitives that are interiors, boundaries and exteriors of A ($A^\circ, \partial A, A^-$) and B ($B^\circ, \partial B, B^-$).

These six topological primitives can be combined to form nine possible combinations representing the topological relationships between these two cells. These 9-intersections are represented as one 3×3 matrix [28]

$$R(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix}$$

The value represented in the matrix will be only a symbol indicating whether the intersection is null (ϕ) or not null ($\neg\phi$). When the value of the intersection is not important, it is represented by ($-$). Based on these nine possible intersections, one can construct 512 theoretical relationships. However, they are not all available. The detection of possible relations is made using negative conditions which prevent the association between pairs of primitives (non-existing topological relations). Therefore, the result implies eight possible topological relations between two regions in $\mathfrak{R}^2$. These eight relations are explicitly represented in Fig. 2 (note that sometimes the MEET relation is called TOUCHES in some papers).

3.2.3. Egenhofer line topological relationship

Egenhofer and Herring define 33 relations can be realized between two simple lines [6]. Fig. 3 shows the different types of intersections and their mathematical interpretations.

3.2.4. Lee and Hsu model

In this model, Lee and Hsu [7,8] study the rectangle relations; they proposed a table representing all spatial relations between two rectangles. They found a total of 169 types (see Fig. 4) in which they number: 48 disjoints, 40 joints, 50 partial overlaps, 16 contains and 16 belongs (=inside). Due to the semantics of ribbons, a lot of them can be discarded. We shall not examine all of them, but the more interesting ribbon relations, namely, disjointing, meeting, merging and crossing.

All the models presented above define topological relationships between objects but they do not treat the transformation of topological relationships between the spatial objects when downscaling. In this paper, we will discuss the transformation of these categories of relationships during the generalization process.

4. Mathematical framework for topological relations

As previously told, it is common to state that there are 0D (points), 1D (lines), 2D (areas) and 3D (solids) geometrical objects for modeling geographical objects. But the reality is much more complex. It is also common to state that streets and rivers can be modeled as lines or polylines, but in reality (ground) they are areas with specific properties so that they can be reduced to lines when needed. In order to take these characteristics into account, the concept of ribbon will be detailed. But before defining a mathematical framework of topological relationships for ribbons and regions, let us present some mathematical background.

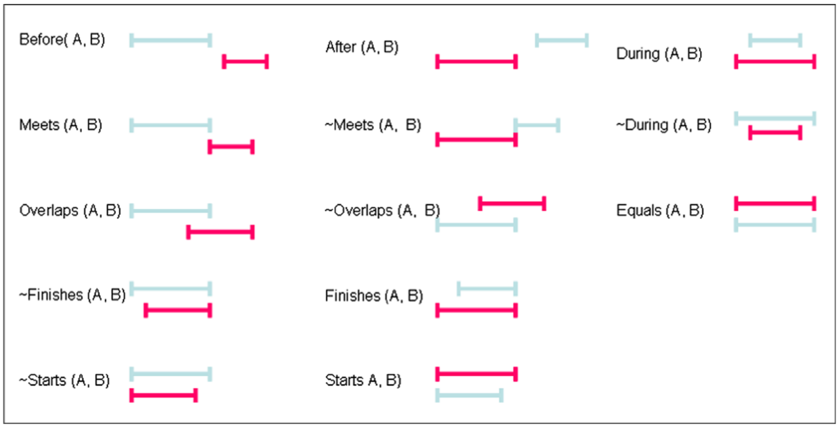

Fig. 1. The Allen topological relations.

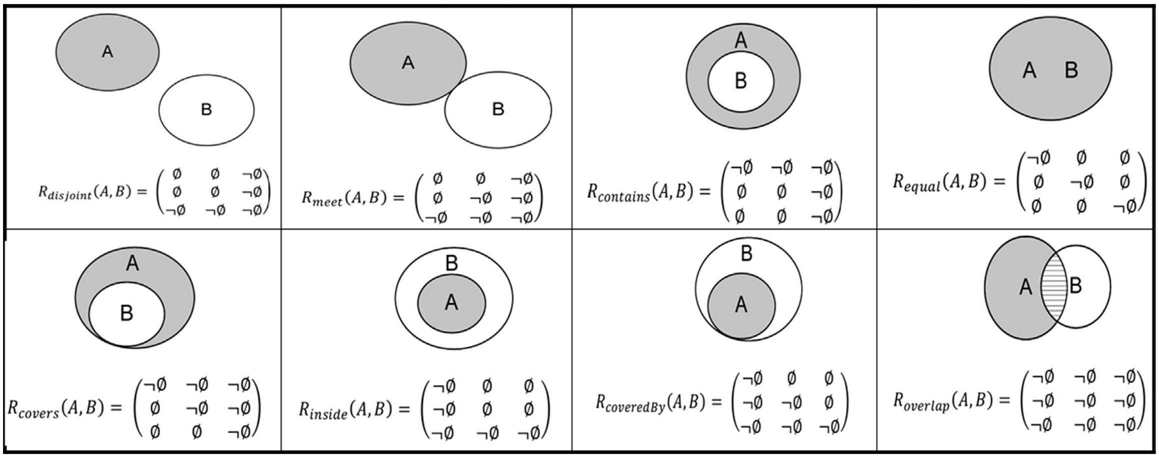

Fig. 2. The eight topological relations between two regions A and B.

4.1. Mathematical background

4.1.1. Definition of ribbon

We claim that ribbons may elegantly model rivers and roads (so-called linear objects): a ribbon can be loosely defined as a line or polyline with a width. Mathematically speaking, a ribbon is defined as a longish rectangle [9]. The ribbon has a skeleton which is its axis. See Fig. 5 for an example.

It is noted that the ribbons have width w, length l and longishness ratio r_l ($r_l=l/w$). The longishness ratio is supposed to be much greater than a positive value r_L so that $r_l > r_L$; a possible minimum value of this threshold r_L is 10.

Let us note Skeleton (R) is the axis of a ribbon. Remember that the ribbon can contain holes which can be useful for modeling islands in rivers.

In the sequel of this paper, to simplify the presentation, a ribbon will be represented by a longish rectangle. For instance, a motorway (see Fig. 6) can be described by several ribbons corresponding to several driving lanes, emergency lanes and one median.

4.1.2. Region

This feature may represent real objects, such as a building. We can define a region as a loose polygonal type. See Fig. 7 for an example, each region has an interior, boundary and exterior. Using these primitives, nine topological relationships can be formed by two regions called the 9-intersection model [6].

4.1.3. Basic theory

In this section, we give certain definitions of the intersection which will be used to formulate the mathematical description for each topological relationship between two ribbons or between a ribbon and a region:

Def 01 # the intersection:

If R^1 and R^2 are two ribbons, to define the intersection of $R^1 \cap R^2$, we have three cases:

- Point P (x,y).
- Line L ($y=ax+b$).
- Area A.

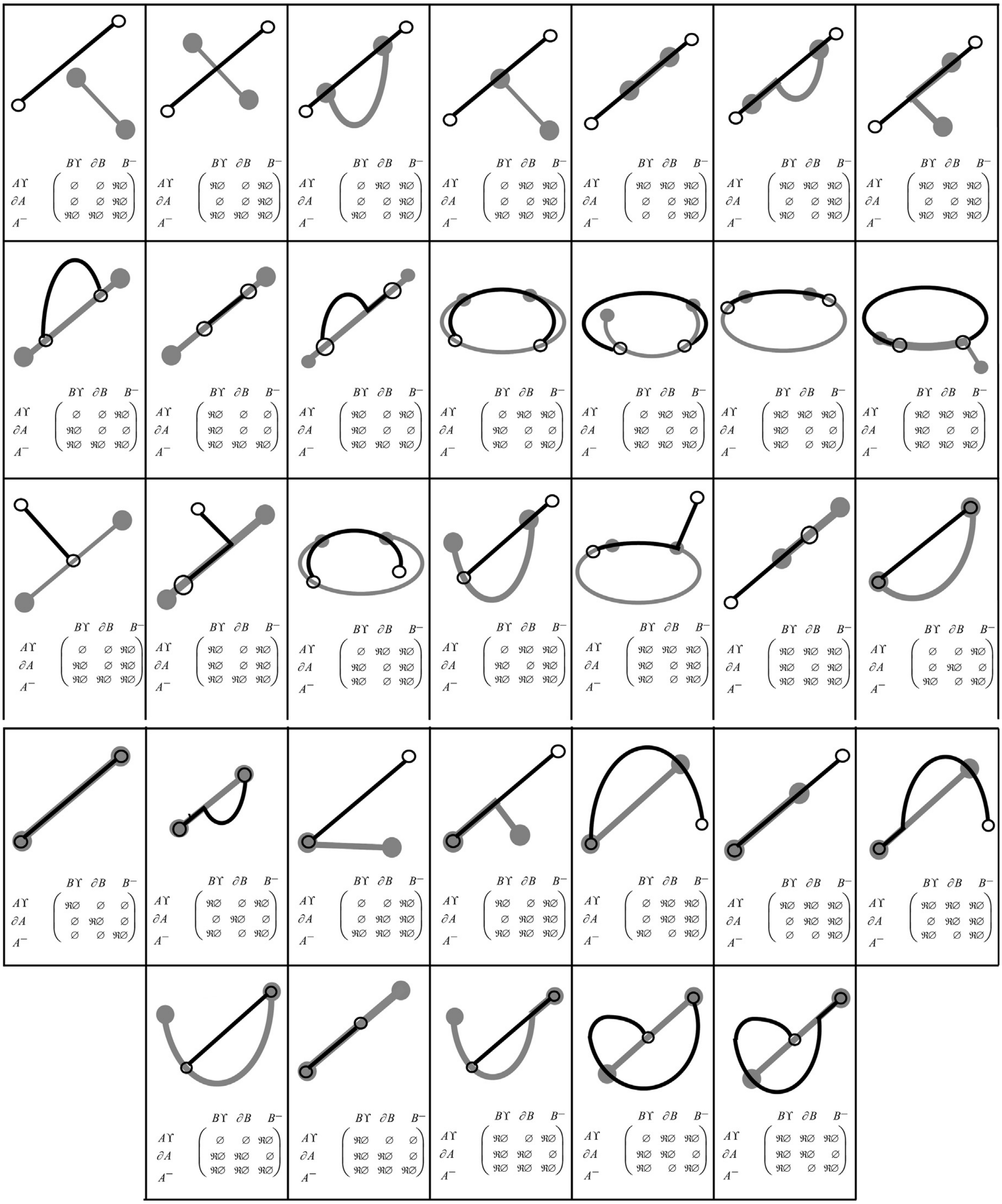

Fig. 3. Topological relations between two lines.

In other terms, this is an exclusive "belong to" defined as follows: $(P \oplus L \oplus A)$. Therefore, we can formulate it as

$$R^1 \cap R^2 = \{x/x \in (P \oplus L \oplus A)\}$$

Def 02 # complement of the intersection:

Let there be two ribbons R^1 and R^2. The **relative complement** of intersection $R^1 \cap R^2$ can be a set of points belonging to R^1 or R^2, but not to$R^1 \cap R^2$. Therefore, we can formally define the relative complement of intersection between ribbons as

$$CMP(R^1 \cap R^2) = \{x/x \in (R^1 \oplus R^2)\ etx \notin (R^1 \cap R^2)\}$$

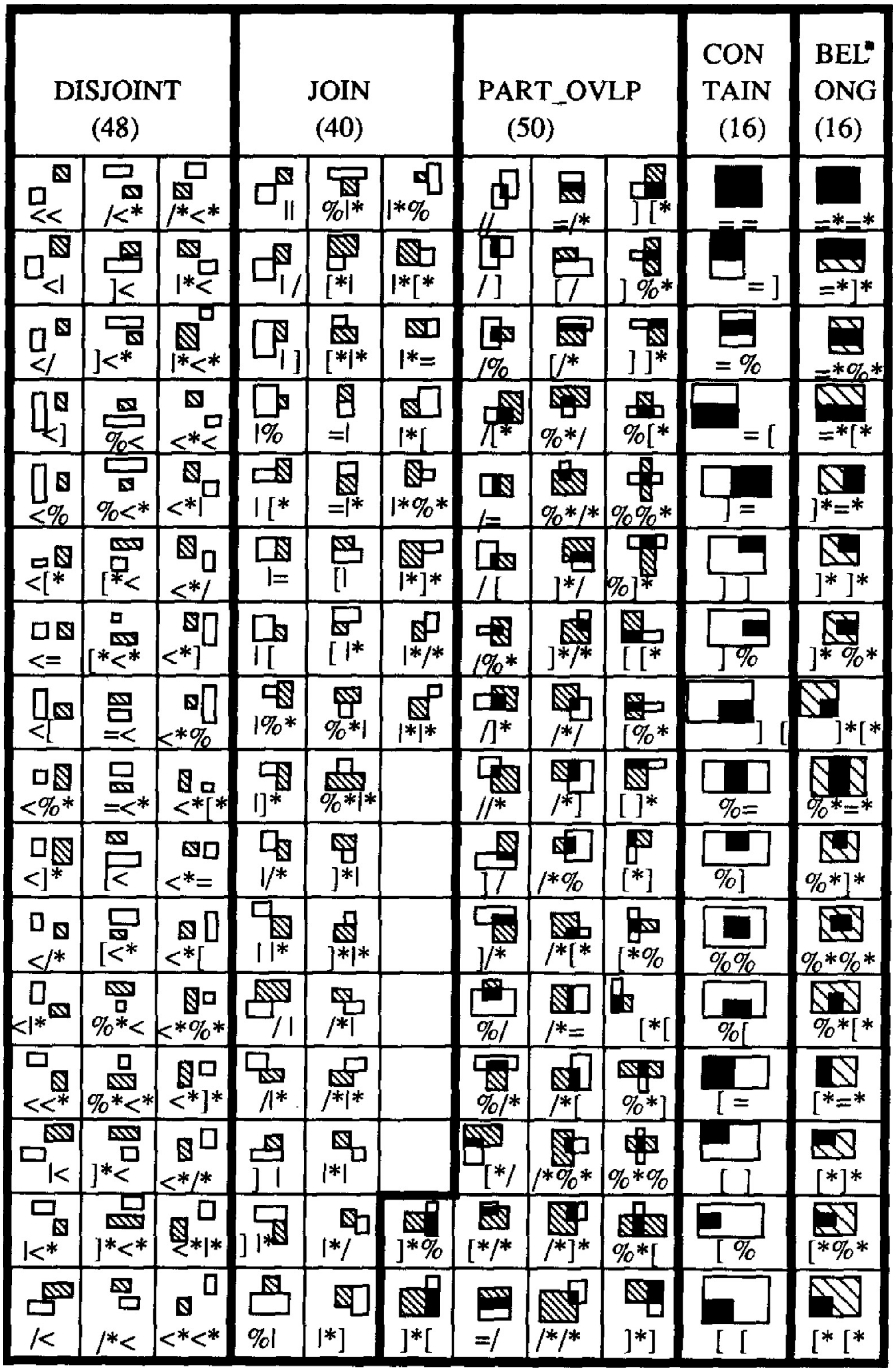

Fig. 4. The 169 types of spatial relations.

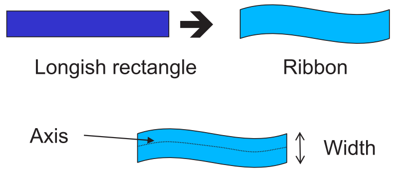

Fig. 5. Definition of ribbon.

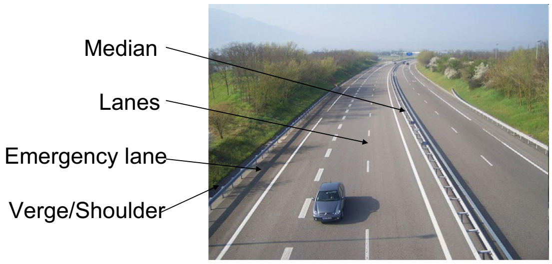

Fig. 6. Ribbon model applied to a motorway.

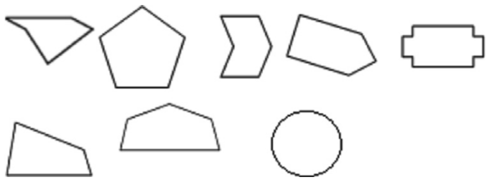

Fig. 7. Example of regions.

4.1.4. Downscaling process

In most cases the required representation scale does not, however, exist in geographical database, thus a derivation from the existing representation of the required representation is necessary. This process of adaptation and reduction of the representation content to a requested scale is called a downscaling process.

During the downscaling, the topological relationships can vary as the changes of objects geometry. We treat in this context, two principal objects: ribbons and regions. We can use the process as it is described in Ref. [9]

Step 0: original geographical features only modeled as areas and/or ribbons,
Step 1: as scale diminishes, small areas and ribbons will be generalized and possibly can coalesce,
Step 2: as scale continues to diminish, areas mutate to points and ribbons into lines (its skeleton), and
Step 3: as scale continues to diminish, points and lines can disappear.

4.1.5. Visual acuity applied to geographical objects

In the GIS, "*Cartographic representation is linked to visual acuity*" [9]. Thresholds must be defined. In classical cartography, the limit ranges from 1 mm to 0.1 mm. If one takes a road and a certain scale and if the transformation gives a width of more than 1 mm, this road is an area, between 1 mm and 0.1 mm it is a line, and if less than 0.1 mm the road disappears. The same reasoning is valid for cities or small countries such as Andorra, Liechtenstein, Monaco, etc. In these cases, the "holes" in Italy or in France disappear cartographically.

In the sequel of this paper, sometimes some of those abbreviations will be used:

$Inters(R^1, R^2)$: represent the intersection between R^1 and R^2;
$Dist(R^1, R^2)$: is the distance between R^1 and R^2;
$Area(R^1 \cap R^2)$: represent the area of the intersection between R^1 and R^2;
$2Dmap(R^1, \sigma)$: is a function transforming a geographical object to some scale possibly with generalization, in the 2-dimension.

Thus, with the defined thresholds ε_i, ε_{lp}, we can formally get

a. Disappearance of a geographical object (O) at scale σ:

$$\forall O \in \text{GeObject}, \forall \sigma \in \text{Scale} \wedge O_\sigma = 2Dmap(O, \sigma) \wedge Area(O_\sigma) < (\varepsilon_{lp})^2 \Rightarrow O_\sigma = \phi.$$

b. transformation of an area into a point (for instance, the centroid of the concerned object, for instance, taken as the center of the minimum bounding rectangle):

$$\forall O \in \text{GeObject}, \forall \sigma \in \text{Scale} \wedge O_\sigma = 2Dmap(O, \sigma) \wedge (\varepsilon_i)^2 > Area(O_\sigma) > (\varepsilon_{lp})^2 \Rightarrow O_\sigma = Centroid(O).$$

c. Transformation of a ribbon R into a line (for instance, its skeleton):

$$\forall R \in \text{Ribbon}, \forall \sigma \in \text{Scale} \wedge R_\sigma = 2Dmap(R, \sigma) \wedge \varepsilon_i > Width(R_\sigma) > \varepsilon_{lp} \Rightarrow R = Skel(R).$$

Therefore, one can say that any spatial relation varies according to scale. As previously told, one says that a road runs along a sea; but in reality, in some place, the road does not run really along the water of the sea due to beaches, buildings, etc. At one scale, the road MEETs the sea (see Fig. 8a), but at another scale at some places, this is a DISJOINT relation (see Fig. 8b). Let us consider two geographical objects O^1 and O^2 and O^1_σ and O^2_σ their cartographic representations, for instance, the following assertion holds:

$$\forall O^1, O^2 \in \text{GeObject}, \forall \sigma \in \text{Scale} \wedge O^1_\sigma = 2Dmap(O^1, \sigma) \wedge O^2_\sigma = 2Dmap(O^2, \sigma) \wedge Disjoint(O^1, O^2) \wedge Dist(O^1, O^2) < \varepsilon_1 \Rightarrow Meet(O^1_\sigma, O^2_\sigma).$$

Similar assertions could be written for CONTAINS, OVERLAP relationships. In addition, two objects in the real world with a MEET relation can coalesce into a single one.

As a consequence, in reasoning what is true at one scale, can be wrong at another scale. So, any automatic system must be robust enough to deal with this issue.

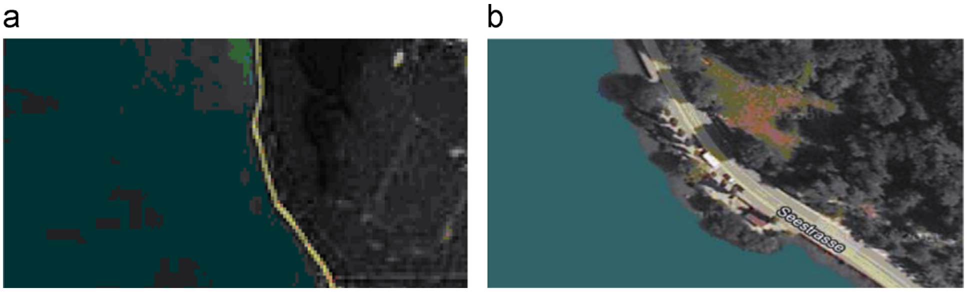

Fig. 8. According to scale, the road MEETs or not the sea.

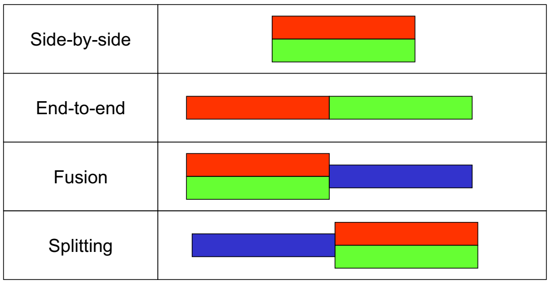

Fig. 9. Basic ribbon relations.

4.2. *Ribbons–ribbons relations*

In a recent paper of Laurini [9], ribbon relations were proposed to describe streets, roads and rivers. Four relations can be defined with ribbons as exemplified in Fig. 9, side-by-side, end-to-end, fusion (or merging) and splitting. For a real world feature (e.g., a road or a river), it can be modeled by a single composite ribbon, that is, a set of ribbons is linked by side-by-side and/or end-by-end relations. As the scale diminishes, ribbons will be reduced to lines, for instance, to their axes ($Axis(R)$). Thus, Laurini [9] has proposed ribbons and partially developed a model for ribbon relationships. In this work, we will complete and refine this model and we will define and classify more topological relationships between ribbons according to certain criteria, then a mathematical description will be given for each type. Thus, two ribbons can be disjoint or intersect. The disjunction is defined by a distance separating the two ribbons. The intersection between two ribbons can be point (0D), line (1D) or area (2D) according to certain criteria. In the following subsection, we will get formally the mathematical description for each topological relationship when we use thresholds and metric measurements; as area, distance, etc. let us present the most important relationships.

4.2.1. *Disjoint relations*

For disjoint relation between two ribbons $Disj(R^1,R^2)$, the first condition is the inexistence of an intersection between them. Fig. 10 shows five cases:

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R^1_\sigma = 2Dmap(R^1, \sigma))$$
$$\wedge (R^2_\sigma = 2Dmap(R^2, \sigma))$$
$$\wedge Inters(R^1, R^2) = \phi \wedge (Dist(R^1, R^2) > \varepsilon_{Ds}) \Rightarrow Disj(R^1_\sigma, R^2_\sigma).$$

4.2.2. *Meeting relations*

Two ribbons R^1 and R^2 are linked by a meeting relation $Meet(R^1, R^2)$ when

the intersection of two ribbons is $P(x,y) \vee L(y = ax + b)$, such as P is Point (0D) and L is Line (1D) (see Fig. 11).

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale})$$
$$\wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma))$$
$$\wedge (Inters(R^1, R^2) = \{P \vee L\}) \wedge (Dist(R^1, R^2)$$
$$= 0) \Rightarrow Meet(R^1_\sigma, R^2_\sigma).$$

4.2.3. *Merging relations*

Two ribbons R^1 and R^2 are linked by a merging relation $Merge(R^1, R^2)$, if the intersection of these ribbons is an area. We obtain six cases, (see Fig. 12):

Formally, we can state:

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale})$$
$$\wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma))$$
$$\wedge Inters(R^1, R^2) \neq \phi$$
$$\wedge (Area(R^1 \cap R^2) > \varepsilon^2_{Mr}) \wedge (Area(CMP(R^1 \cap R^2)) = 0)$$
$$\Rightarrow Merge(R^1_\sigma, R^2_\sigma).$$

4.2.4. *Crossing relations*

This topological relationship is very important because 80% of spatial objects are polyline-type [31]. Common examples include road–road crossings and river–road crossings. For instance, see Fig. 13.

This relation is based on the area of the intersection between two ribbons R^1 and R^2. For instance, a threshold ε_{Cr} can be given.

So, we have

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale})$$
$$\wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma))$$
$$\wedge Inters(R^1, R^2) \neq \phi$$
$$\wedge (Area(R^1 \cap R^2) > \varepsilon_{Cr}) \wedge (Area(CMP(R^1 \cap R^2)) > 0)$$
$$\Rightarrow Cross(R^1_\sigma, R^2_\sigma).$$

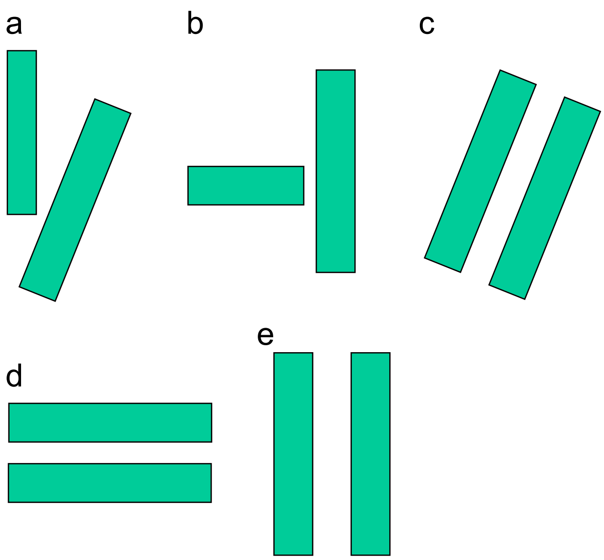

Fig. 10. Disjoint relations between two ribbons.

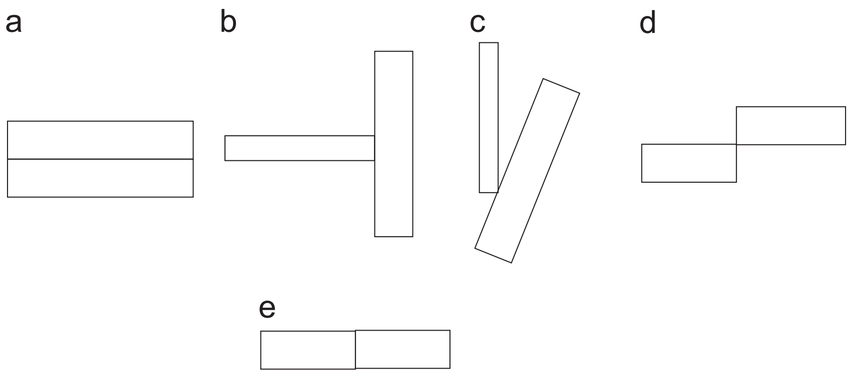

Fig. 11. Several cases for meeting from (b–d). Except (a) corresponding to a side-by-side and (e) to end-to-end.

4.3. Transformation of ribbons–ribbons relations

When downscaling, the transformation of topological relations can be applied. The topological relations between objects varied according to certain criteria, we present in the following subsection these transformations.

4.3.1. Transformation of disjoint to merge

This disjoint relation transformed into merging relation, when downscaling (see Fig. 14).

This process can be modeled as follows:

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma)) \wedge Disj(R^1, R^2) \wedge (Dist(R^1, R^2) < \varepsilon_{Dj}) \Rightarrow Merg(R^1_\sigma, R^2_\sigma).$$

When a ribbon becomes very narrow, we apply this assertion:

$$\forall R \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R_\sigma = 2Dmap(R, \sigma)) \wedge (Width(R_\sigma) < \varepsilon_{lp}) \Rightarrow R_\sigma = \phi.$$

4.3.2. Transformation of cross to merge

The crossing relation can transform into merging relation according to the area of complement of the intersection between the two ribbons: $Area(CMP(R^1 \cap R^2))$, (see Fig. 15).

The formal definition of this process is

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma)) \wedge (Cross(R^1, R^2)) \wedge (Area(CMP(R^1 \cap R^2)) < Area(R^1 \cap R^2)) \Rightarrow Merge(R^1_\sigma, R^2_\sigma).$$

When a ribbon becomes very narrow, we apply this assertion:

$$\forall R \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R_\sigma = 2Dmap(R, \sigma)) \wedge (Width(R_\sigma) < \varepsilon_{lp}) \Rightarrow R_\sigma = \phi.$$

4.3.3. Transformation of meet to merge

The transformation of meeting relation to merging relation was expressed by the following assertion

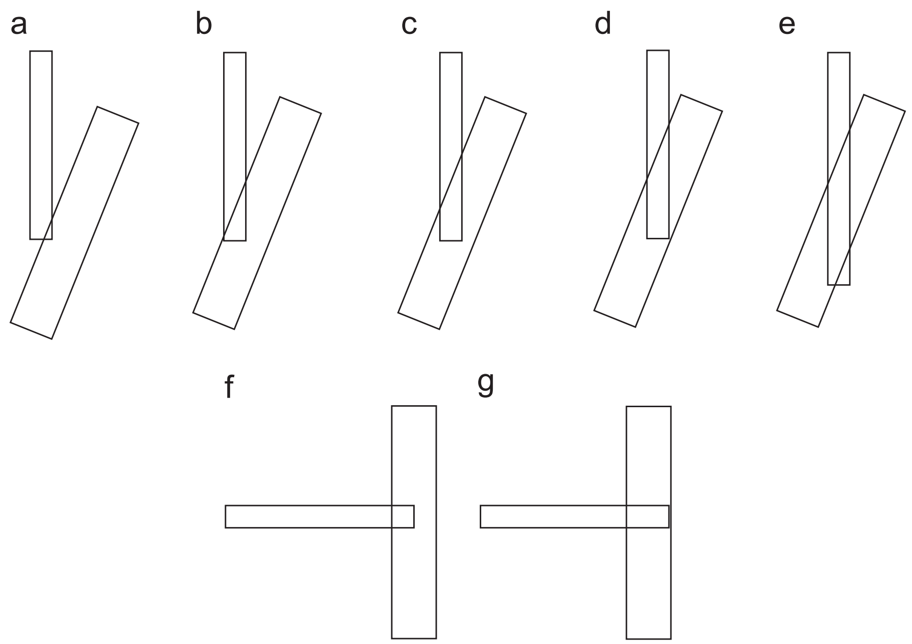

Fig. 12. Example of merging.

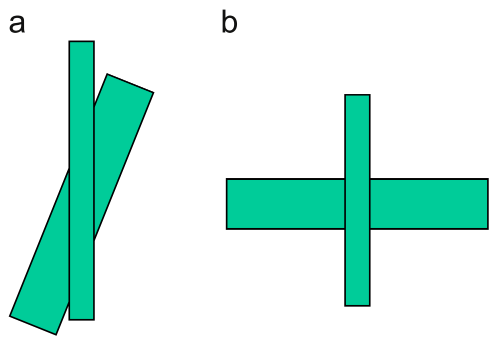

Fig. 13. Example of crossing.

(see Fig. 16):

$$\forall R^1, R^2 \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R^1_\sigma = 2Dmap(R^1, \sigma)) \wedge (R^2_\sigma = 2Dmap(R^2, \sigma)) \wedge (meet(R^1, R^2)) \wedge (Area(R^1 \cap R^2)) > \varepsilon^2_{Mr}) \Rightarrow merge(R^1_\sigma, R^2_\sigma).$$

When a ribbon becomes very narrow, we apply this assertion:

$$\forall R \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R_\sigma = 2Dmap(R, \sigma)) \wedge (Width(R_\sigma) < \varepsilon_{lp}) \Rightarrow R_\sigma = \phi.$$

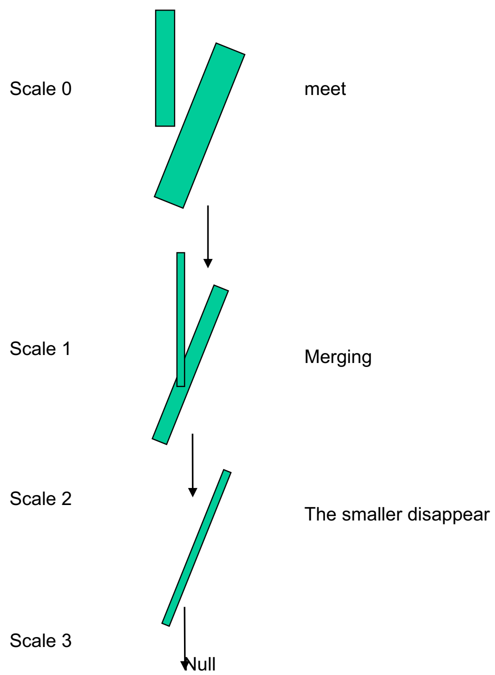

Fig. 14. Transformation of disjoint relation between two ribbons.

4.4. Ribbons–regions relations

In this section, we study the relations which can hold between ribbons and regions. To describe these relations, we can classify them into types, namely, disjoint, meet or touches, cross, cover (covered-by), contain (contained by), overlap and on-boundary, as shown in Fig. 17.

The common example in this case is when a road runs along the sea, what are exactly the spatial or geographical relations which are concerned? Sometimes, either the road touches the sea or a small beach is located between the road and the sea, etc. From a mathematical point of view, mostly there is a disjoint relation between the road and the sea whereas for people the relation is different.

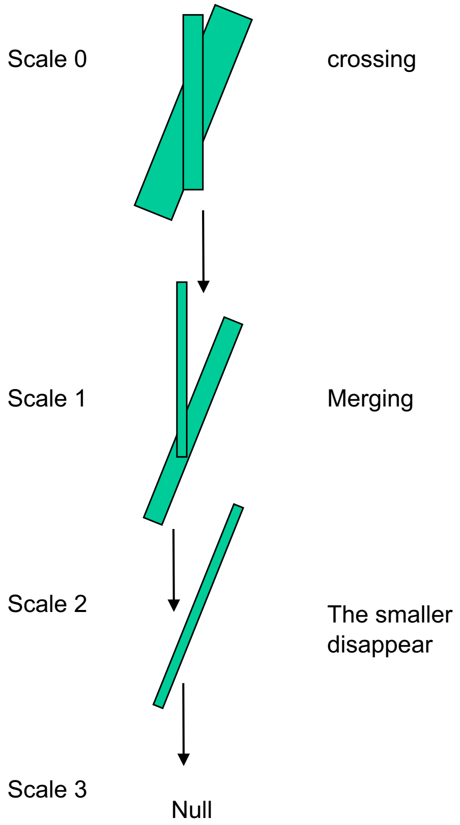

Fig. 15. Transformation of crossing relation between two ribbons.

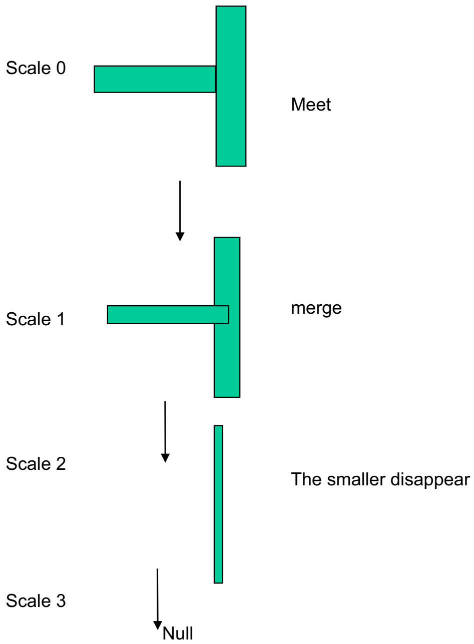

Fig. 16. Transformation of merge relation to meet.

In addition, when one is reading a map, according to scale, the topological relation can be different, disjoint or meet.

Also, the relations between regions and ribbons can also be varied according to the scale, for instance, the disjoint relation may be transformed into meet or on-boundary relations when downscaling.

4.5. Transformation of ribbons–regions relations

4.5.1. Transformation of disjoint to meet

The disjoint relation can transform into relation meet according to the following conditions (see Fig. 18):

The exact condition is that the distance between region and ribbons must be greater than the thresholds ε_{Ds}.

This transformation can be applied according to this assertion:

$$\forall R \in \text{Ribbon}, \forall G \in \text{Region}, (\forall \sigma \in \text{Scale}) \wedge (R_\sigma = 2Dmap(R,\sigma)) \wedge (G_\sigma = 2Dmap(G,\sigma)) \wedge Disj(R,G) \wedge (Dist(R,G) > \varepsilon_{Ds}) \Rightarrow Meet(R^1_\sigma, R^2_\sigma).$$

When a ribbon becomes very narrow, we apply this assertion:

$$\forall R \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (R_\sigma = 2Dmap(R,\sigma)) \wedge (Width(R_\sigma) < \varepsilon_1) \Rightarrow R_\sigma = \phi.$$

The region can be eliminated if its area is too small to be well visible. Thus, in this case, the initial relation does not hold anymore.

$$\forall O \in \text{GeObject}, \forall \sigma \in \text{Scale} \wedge (O_\sigma = 2Dmap(O,\sigma)) \wedge (Area(O_\sigma) < (\varepsilon_{lp})^2) \Rightarrow O_\sigma = \phi.$$

4.5.2. Transformation of contain to cover

The transformation of contain relation into cover relation was expressed by the following assertion (see Fig. 19):

$$\forall G \in \text{Region}, \forall R \in \text{Ribbon}, (\forall \sigma \in \text{Scale}) \wedge (G_\sigma = 2Dmap(O,\sigma)) \wedge (R = 2Dmap(R,\sigma)) \wedge (Contains(R, G)) \wedge (Dist(R, G) < \varepsilon_1) \Rightarrow Cover(O^1_\sigma, O^2_\sigma).$$

But a smaller object can disappear or be eliminated if its area is too small to be well visible. So in this case, the initial relation does not hold anymore.

4.5.3. Generalized irregular tessellations when downscaling

By irregular tessellation (or tessellation), one means the total coverage of an area by sub-areas. For instance, the conterminous States in the USA form a tessellation to cover the whole country. Generally speaking, administrative subdivisions form tessellations, sometimes as hierarchical tessellations. Let us consider a domain D and several polygons P_i; they form a tessellation if (see Fig. 20b):

- For any point p_k, if p_k belongs to D then there exists P_j, so that p_k belongs to P_j.
- For any p_k belonging to P_j, then p_k belongs to D.

A tessellation can also be described by Egenhofer relations applied to P_i and D, but in practical cases, due

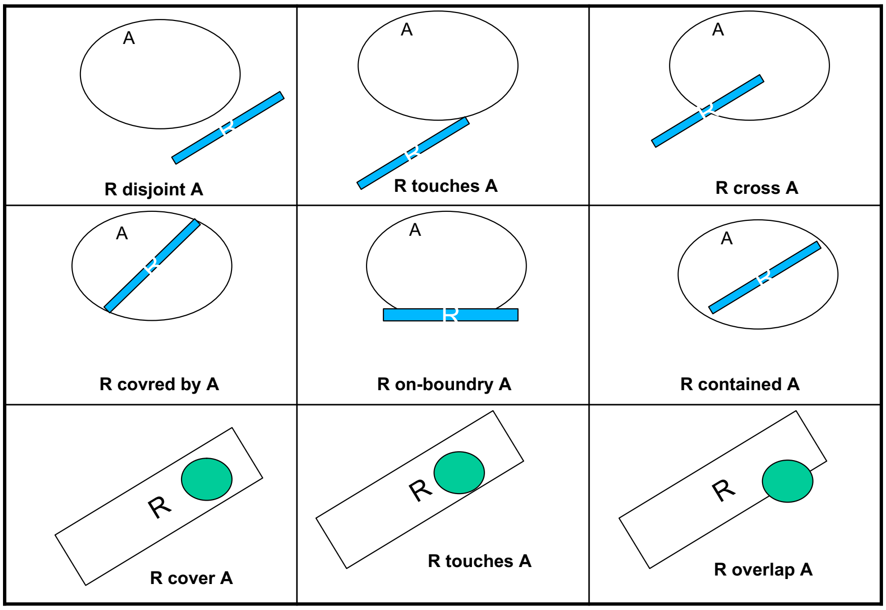

Fig. 17. Basic relations between regions and ribbons.

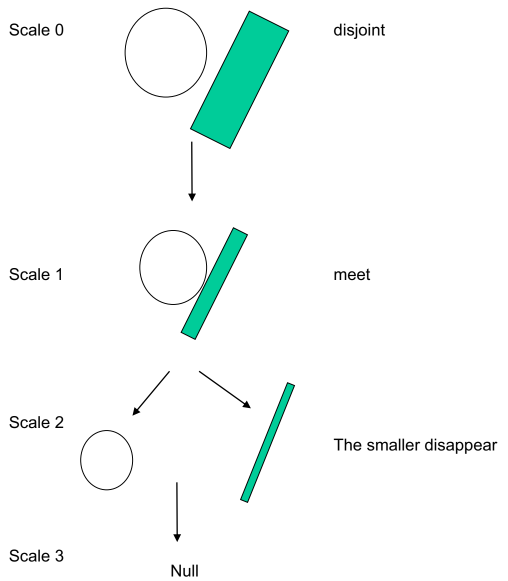

Fig. 18. Transformation of disjoint into meet.

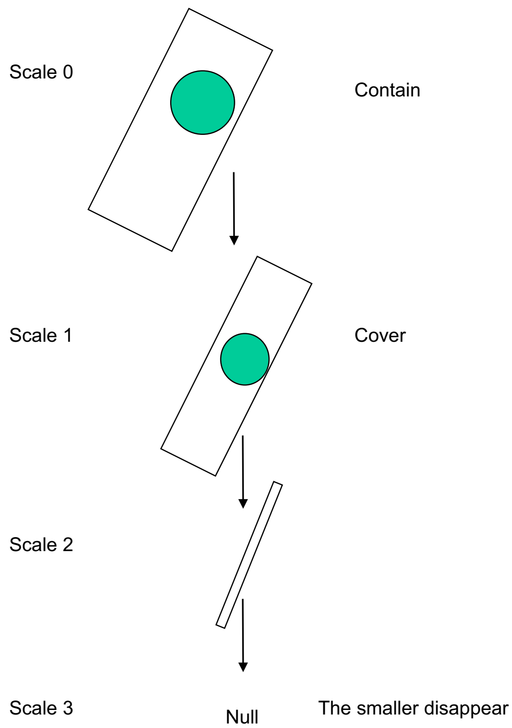

Fig. 19. Transformation of contain into cover.

to measurement errors, this definition must be relaxed in order to include sliver polygons (see Fig. 20a). Those errors are often very small, sometimes a few centimeters at scale 1. In other words, one has a tessellation from an administrative point of view, but not from a mathematical point of view.

When downscaling, those errors will be rapidly less than the threshold ε_{lp} so that the initial slivered or irregular tessellation will become a good-standing tessellation.

The situation becomes complex when roads or rivers traverse the tessellation, because we have to study all topological relationships between tessellation and ribbons which represent the roads or rivers.

4.6. Chain of ribbons

A chain is defined by a set of ribbons linked by end-to-end relations. A ribbon chain may be closed; in this case, it

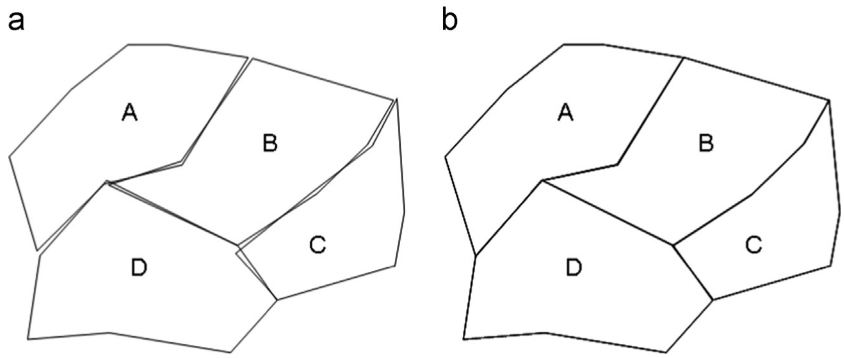

Fig. 20. A tessellation with sliver polygons and a good standing tessellation.

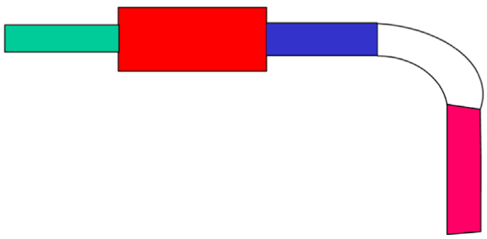

Fig. 21. Chain of ribbons.

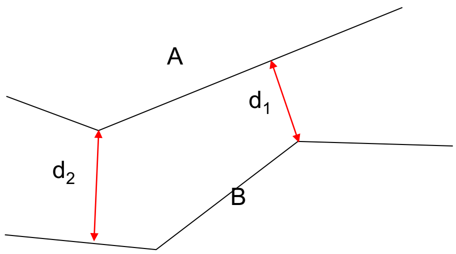

Fig. 22. The distance between two polylines.

constitutes a ribbon circuit. In general, since river's and road's widths are variable, they can be modeled by a chain of ribbons (see Fig. 21).

4.7. Distance between objects

We concentrated, in this work, especially on the distance between objects. Considering two objects *A* and *B*, what is the distance between them? An interesting definition is given by the Frechet distance which corresponds to the minimum leash between a dog and its owner, the dog walking on a line, and the owner in the other line as they walk without backtracking along their respective curves from one endpoint to the other. The definition is symmetrical with respect to the two curves (see Fig. 22) [11]. By noting *a*, a point of *A*, and *b* of *B*, the Frechet distance *F* is given as follows in which *dist* is the Euclidean conventional distance:

$$F = \underset{a \in A}{Max}(\underset{b \in B}{Min}(dist(a, b)))$$

But in our case, one must consider two distances, let us say, the minimum and the maximum of the leash, so giving:

$$d1 = \underset{a \in A}{Min}(\underset{b \in B}{Min}(dist(a, b)))$$

and

$$d2 = \underset{a \in A}{Max}(\underset{b \in B}{Min}(dist(a, b))).$$

The thresholds used in the mathematical assertions are defined from this distance. Then, the distance between two regions *A* and *B* is defined also as the Frechet distance between both boundaries. In this context, the algorithm defined in [11] is used.

4.8. Experimental analysis

The downscaling of a map implies that the topological relationships between spatial objects (ribbons and regions) should be transformed into other ones. To apply these transformations on the map, the following requirements should be satisfied: (1) the topological relationships between spatial objects should be defined, (2) a framework is required to derive the transformations of topological relationships, and (3) some metric measures and thresholds are taken to guide these transformations.

To analyze the variation of topological relationships using our mathematical framework, a prototype is developed. We implement three main functions

- *Dist* (O^1,O^2): calculate the distance between two objects;
- *RelTOPO*(O^1,O^2): define the relations holding between two objects;
- *TRansRel*(Rel, Dist, threshold): apply the possible transformations using the mathematical assertions.

Using predefined thresholds, the prototype works as the following steps:

- Compute and store the topological relationships between the ribbons and/or regions, using the assertions developed in Sections 4.2 and 4.4.
- Apply the simplification operator of the generalization process.
- Use the assertions developed in Sections 4.3 and 4.5 to transform the possible topological relationships

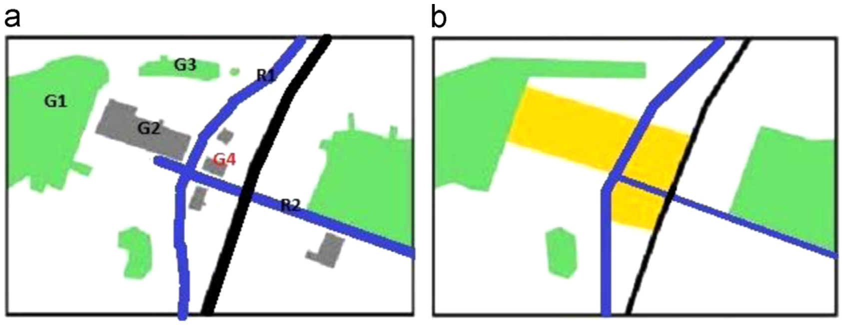

Fig. 23. Transformation of topological relationships. (a) before generalization, (b) after generalization.

Table 1
The variation of topological relationships.

Before generalization	After generalization
Disjoint(G^1, G^2)	Meet (G^1, G^2)
Cross(R^1, R^2)	Merge (R^1, R^2)
Disjoint(G^1, G^3)	Meet(G^1, G^3)
Disjoint (R^2, G^4)	Meet(R^2, G^4)

between the simplified objects (regions and ribbons).

The prototype can automatically detect the topological relationships between objects and transform them into other ones according to the mathematical assertions described in Sections 4.3 and 4.5.

Fig. 23 shows a real example; the River R^1 is crossed with another river R^2 and the buildings (G^1, G^2 and G^3) are disjointed. The topological relationships are as follows:

Disjoint(G^1, G^2).
Cross(R^1, R^2).
Disjoint(G^1, G^3).
Disjoint (R^2, G^4).

Let us define two thresholds, ε_i for invisibility of objects and ε_{pl} for the reduction of objects (regions or ribbons) to point or line. In our implementation, we take$\varepsilon_i =$ 0.1 mm and $\varepsilon_{lp} = 1$ mm. When downscaling, the rivers and the buildings are generalized and the topological relationships are transformed into other relationships. Table 1 illustrates these transformations of topological relationships

Fig. 24 shows another real example, consisting of several objects: rivers, coast, city. Thus, the French Riviera coast runs along the Mediterranean Sea, from Spain to Italy. There are three cities: Nice, Montpelier and Marseilles. The Rhone River is linked to the sea.

Meet (River, Coast).

When downscaling, the Mediterranean coast is generalized and the topological relation is transformed into

Merge (River, Coast).

Since certain topological relations must be persistent, regardless of the scale of representation, those relations must hold. See, for instance, in Fig. 24 the Mediterranean Coast in the South of France: as the coast is generalized, some harbors will be in the middle of the sea such as Nice, whereas others will be inside the country such as Marseilles and Montpellier; in addition, the confluence of the Rhone river will be badly positioned in the middle of the land. The constraints are as follows:

Covers (France, Nice).
Covers (France, Marseilles).
Covers (France, Montpellier).
Covers (France, Rhone).

Another example of topological constraint when generalizing the Eastern French border is the case of Geneva which must hold outside France (see Fig. 25); the constraint is as follows:

Meet (France, Geneva).

In our implementation, we use the ribbons to represent the linear objects in order to verify the correctness of the concepts of the proposed framework. In this study, we present some examples to show the transformations of topological relationships when downscaling. The topological consistencies of the map are required when downscaling. However, traditional methods for maintaining consistencies of topological relationships are ineffective as they do not associate the shape simplification with the transformation of topological relationships. Thus, they cannot analyze the transformations of topological relationships; this makes them ineffective and weak to preserve topological consistencies in the map.

The framework presented in this paper consists of transforming the topological relationships into other ones in order to maintain the consistencies of topological relationships, thus, keep the high quality of the map when downscaling. Our collection of the cases we tested in the three previous examples corresponds to different topological relationships between spatial objects (ribbons or regions) as disjoint, meet, cross, merge and covers. They have been successfully tested and indicate the correctness of our concepts and the ability of our mathematical assertions to transform the topological relationships from any given map.

This study focuses only on the transformation of topological relationships when downscaling. The mathematical assertions of this framework can be integrated on any simplification algorithm provided by GIS as the algorithm presented in Ref. [15], but this is beyond the scope of this study. This work will be addressed in the future.

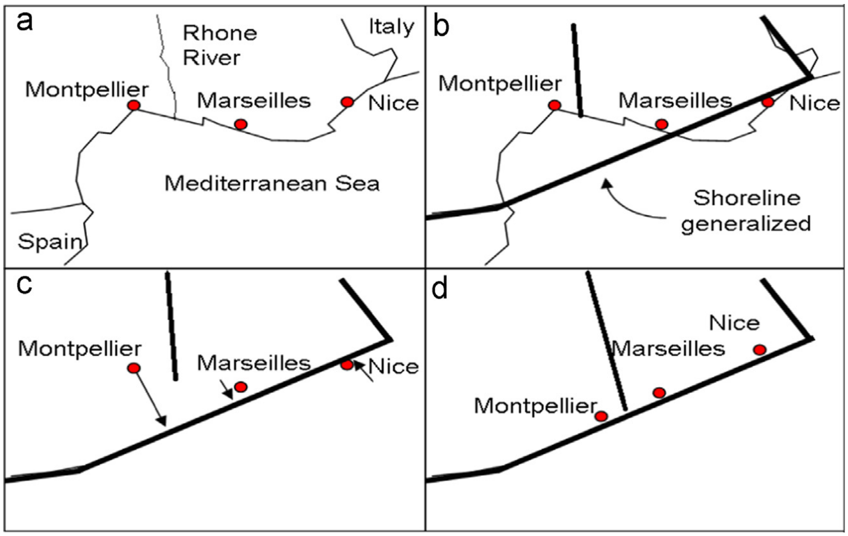

Fig. 24. Holding topological constraints for harbors in the Mediterranean Sea. (a) Before generalization. (b) Only the coastline is generalized. (c) Harbors must move. (d) After generalization (the meet relation transformed into merge).

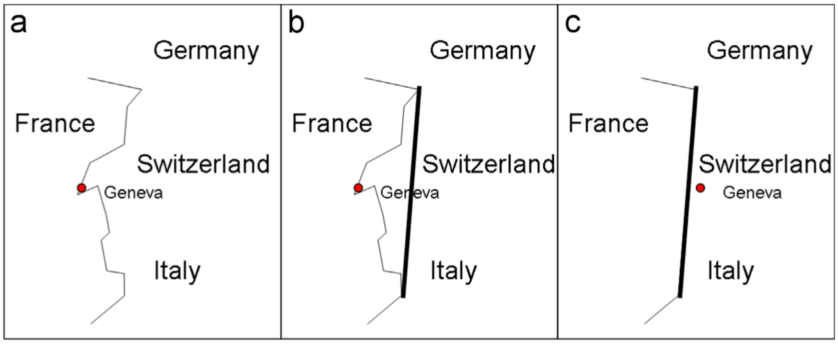

Fig. 25. Holding topological constraints for outside border cities.

5. Conclusion and future works

The concept of generalization was introduced into the GIS domain many years ago. Many propositions were given for modeling generalization but propositions do not exist which treat really the topological relations issues.

The application of the generalization operators may cause topological conflicts. To avoid these conflicts, topological conditions are used to generate the relationships in terms of meeting, overlapping, disjunction, and containment between map objects into others relationships. In this paper, we use these topological conditions to formulate some of mathematical frameworks which are composed of a set of assertions for treating the variety of topological relations according to the scale. We consider two principal types of objects: regions and ribbons. When downscaling, a spatial object, represented by area, can mutate into a point, or disappear; also a ribbon can mutate into a line, or disappear. These objects have topological relationships between them. So, each topological relation will also be generalized using the assertions given in mathematical framework for each situation. This framework was based on three principle models of relations, Allen [5], Egenhofer [6] and Lee and Hsu [7,8].

This work can open various future works, such as:

- Integration of this topological model in on-the-fly web map generation.
- A map does not contain only the simple objects such as areas and ribbons, but there are also roundabouts and motorway interchanges that are complex objects. A complementary mathematical framework for this type of objects will be a future work.
- The assertions of the mathematical framework considered the geometries of object represented in the 2D domain, we would like to extend our work to deal with geometries of higher dimension, such as the 3D.

And finally, the foundations of a robust topology with newly presented concepts of ribbons, loose ribbons, chain of ribbons and their particular relations can be more useful to help solve real problems in geographical reasoning and in territorial intelligence. Indeed we need to design a theory which must be robust against measurement errors and downscaling.

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