Local convexity properties of digital curves

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Abstract

The paper studies local convexity properties of digital curves. We locally define convex and concave parts from the slope of maximal digital straight segments and arithmetically characterize the smallest digital pattern required for checking convexity.

Moreover, we introduce the concepts of digital edge and digital hull, a digital hull being a sequence of increasing or decreasing digital edges. We show that any strictly convex or concave part has a unique digital hull and arithmetically characterize the smallest digital pattern that contains a point incident to two consecutive digital edges.

These theoretical results lead to online and linear-time algorithms that are useful for polygonal representations: convex hull computation of convex digital curves, computation of a reversible polygon that respects the convex and concave parts of a digital curve and computation of the well-known minimum-perimeter polygon.

1 Introduction

In this paper we consider the local convexity properties of digital curves, *i.e.* paths of simply 4-connected lattice points.

The problem that consists in decomposing objects into perceptually meaningful parts is of great interest in shape recognition and is closely related to convexity. The concept of convexity plays an important role since almost all visible objects are either convex or else composed of a finite number of convex objects. A visual part is not necessarily convex but it can be assumed that it is convex at a given scale [1]. Thus, finding a good scale with respect to a given purpose or studying the convexity at various scales may solve the problem.

In addition, the convex and concave parts of an object straightforwardly give the sign of the curvature of its boundary, which is known to be a relevant piece of information about its shape. For instance, curvature zero-crossings at various scales leads to an interesting shape representation, known as the curvature scale space representation [2]. Moreover, at a given scale, the endpoints of maximal convex (resp. concave) parts correspond to points of minimal negative (resp. maximal positive) curvature [1]. Convexity is thus also a way to retrieve some dominant points.

In Euclidean geometry, a given region $\mathcal{R}$ is said to be convex if and only if for any pair of points $p, q \in \mathcal{R}$ the line segment $[pq]$ is included in $\mathcal{R}$. However, in digital image processing, when each pixel is viewed as a point of $\mathbb{Z}^2$, the only convex regions (in the Euclidean sense) are isolated points, which is not satisfactory at all. Many authors defined the convexity of digital sets, *i.e.* sets of points of $\mathbb{Z}^2$ (see for instance, Sklansky [3], Kim [4, 5], Kim and Rosenfeld [6, 7], Kim and Sklansky [8], Chassery [9] and Ronse [10]). Most of these definitions may be proved to be equivalent for simply connected sets [5, 4, 10, 11]. However, they fail to properly define the convex and concave parts of open digital curves (because a convex open digital curve may also be considered as a digital set that is not convex). In many applications such as line drawings processing, dealing with open digital curves is important. That is why we define in this paper convex and concave parts by means of the slope of the maximal digital straight segments, *i.e.* digital straight segments of a digital curve that cannot be extended at the front or at the back (see [12] for a review about digital straightness). This definition was first proposed by [13] and is also used in [14, 15]. In a practical point of view, this definition enables us to deal with any digital curve, which may correspond to the boundary of a given digital region or not, like digital spirals (fig. 4). In a theoretical point of view, this definition appears to be quite natural since convexity is closely related to straightness: convex sets are defined by means of line segments and line segments are convex sets.

Deciding whether a given part of a digital curve is convex or not is not a trivial task. As shown in [11], the convexity of the boundary of a digital re-

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gion cannot be decided locally (where locally means in the 8-neighborhood). Considering this fact, the following question has been raised in [16]: how far one can decide whether a part of a digital boundary is convex or not by a method that is *as local as possible*? Even though some clues may be found in [17][Theorem 9] and [18][Lemma 1], no full answer has been given yet. In this paper, we answer to this question and go further by investigating related problems about local convexity properties of digital curves.

In section 2, we recall some basic definitions, tools and results of digital geometry. More precisely, we give an arithmetic definition of the digital straight segments in definition 1, we introduce the cover of a digital curve by maximal segments in definition 4 and prove some important results about the intersection between consecutive maximal segments in lemmas 1, 2 and corollary 1.

In section 3, we locally define convex and concave parts from the slope of maximal segments in definition 5 and prove theorem 1 and 2 from the results of the previous section about the intersection between consecutive maximal segments. These theorems lead to an arithmetic characterization of the smallest digital pattern required for checking convexity and to algorithm 1, an online and linear-time algorithm dedicated to the decomposition of a digital curve into convex and concave parts.

In section 4, we introduce the concepts of digital edge in definition 7 and digital hull in definition 8, a digital hull being a sequence of increasing or decreasing digital edges. After proving lemmas 3 and 4, we show theorem 3 that states that any strictly convex or concave part has a unique digital hull. Theorem 4 arithmetically characterizes the smallest digital pattern that contains a point incident to two consecutive digital edges and lead to algorithm 2, an online and linear-time algorithm devoted to the computation of the digital hull of a strictly convex or concave digital curve.

We show in section 5 that these results are useful for polygonal representations: convex hull computation of convex digital curves (section 5.1), computation of a reversible polygon that respects the convex and concave parts of a digital curve (section 5.2) and computation of the well-known minimum-perimeter polygon (section 5.3).

2 Preliminaries

We work below with the concept of 4-neighborhood. The results derived in the rest of the paper are applicable for 8-neighborhood but require more complicated proofs and algorithms.

The x-coordinate and y-coordinate of a point $P \in \mathbb{Z}^2$ are respectively defined as maps $x : \mathbb{Z}^2 \to \mathbb{Z}$ such that $x(P)$ is the x-coordinate of P and $y : \mathbb{Z}^2 \to \mathbb{Z}$ such that $y(P)$ is the y-coordinate of P.

The vector starting from a point $P \in \mathbb{Z}^2$ and ending at a point $Q \in \mathbb{Z}^2$ is denoted by $\vec{PQ}$ and is equal to $Q - P$, *i.e.* the x-component of $\vec{PQ}$, conveniently denoted by $x(\vec{PQ})$, is equal to $x(Q) - x(P)$ and the y-component of $\vec{PQ}$, conveniently denoted by $y(\vec{PQ})$, is equal to $y(Q) - y(P)$. The city-block norm of a vector $\vec{PQ}$ is denoted by $\| \vec{PQ} \|_1$ and is equal to $(|x(\vec{PQ})| + |y(\vec{PQ})|)$.

Two points $P, Q \in \mathbb{Z}^2$ are 4-neighbors if and only if $\| \vec{PQ} \|_1 = 1$.

Following Rosenfeld's definition of simple 4-path [19], a digital curve, denoted by C, is a sequence of points $C_1, C_2, \ldots, C_n \in \mathbb{Z}^2$ such that for all $i \in 1, \ldots, n-1$, C_i has exactly two 4-neighbors, which are C_{i-1} and C_{i+1}.

Moreover, C is open if C_1 and C_n have only one 4-neighbor, and closed if C_1 and C_n also have exactly two 4-neighbors, which are respectively C_n and C_2, C_{n-1} and C_1.

Any subsequence $C_i, C_{i+1}, \ldots, C_{j-1}, C_j$ such that $1 \leq i \leq j \leq n$, conveniently denoted by $C_{i|j}$, is an open digital curve.

We will study below the local straightness and convexity properties of such open digital curves. Closed digital curves will be processed as open ones in section 5.

2.1 Digital Straight Segment

The following definition of digital straight segment stems from the arithmetic definition of digital straight line [20]:

Definition 1 (Digital straight segment) *A digital curve $C_{i|j}$ is a digital straight segment (DSS) if and only if there exists $p, q, l \in i, \ldots, j$, with $p < q$, such that $\vec{C_pC_q}$ has two relatively prime components,* i.e. $\gcd(x(\vec{C_pC_q}), y(\vec{C_pC_q})) = 1$, *and for all $k \in i, \ldots, j$, $0 \leq det(\vec{C_pC_q}, \vec{C_lC_k}) < \| \vec{C_pC_q} \|_1$.*

The well-known chord property of Rosenfeld [21] may be seen as a Helly type property of definition 1 [22]. Moreover, each DSS is a connected part of the digital straight line $D(a, b, \mu, \omega)$ of Reveillès [20] where $a = -y(\vec{C_pC_q})$, $b = -x(\vec{C_pC_q})$, $\mu = a.x(C_l) - b.y(C_l)$ and $\omega = |a| + |b|$.

The formalism of definition 1 has been introduced in order to highlight the existence of a determinant calculus and to directly deal with segments (instead of lines).

Let $C_{i|j}$ be a DSS and let $p, q, l \in i, \ldots, j$ be such that the conditions of definition 1 are fulfilled. We introduce below some related definitions illustrated in fig. 1.

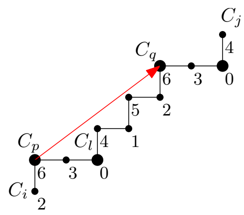


Figure 1: Example of a DSS of direction vector $u(C_{i|j}) = \overrightarrow{C_pC_q} = (3,4)$. The remainders of the points are between 0 and $\|u(C_{i|j})\|_1 = 6$. The lower and upper leaning points have a remainder respectively equal to 0 and 6, C_l being one of the two lower leaning points.

The *direction vector* of $C_{i|j}$, denoted by $u(C_{i|j})$, is defined by $u(C_{i|j}) = \overrightarrow{C_pC_q}$.

The *remainder* of a point $M \in \mathbb{Z}^2$ with respect to $C_{i|j}$, denoted by $r(C_{i|j}, M)$, is equal to the determinant of $u(C_{i|j})$ and $\overrightarrow{C_lM}$, *i.e.* $r(C_{i|j}, M) = det(u(C_{i|j}), \overrightarrow{C_lM})$.

Let us consider two vectors $\vec{u}, \vec{v}$ such that the components of the first one are relatively prime, *i.e.* $\gcd(x(\vec{u}), y(\vec{u})) = 1$. Due to basic arithmetic results, the equation $det(\vec{u}, \vec{v}) = r$, when $r \in \mathbb{Z}$, has infinitely many solutions belonging to the following set: $\{\vec{v} + z. \vec{u} \mid z \in \mathbb{Z}\}$. In other words, the points of $C_{i|j}$ having equal remainders with respect to $C_{i|j}$ lie on a straight line of slope $\frac{y(u(C_{i|j}))}{x(u(C_{i|j}))}$ ($\frac{3}{4}$ in fig. 1). Moreover, between two points of index $p, q \in i, \dots, j$ such that $\overrightarrow{C_pC_q} = u(C_{i|j})$, there is no pair of points with the same remainder. Since there are exactly $\|u(C_{i|j})\|_1 - 1$ points within the range $p, \dots, q-1$ and exactly $\|u(C_{i|j})\|_1 - 1$ different values of remainder within the range $0, \dots, \|u(C_{i|j})\|_1 - 1$, there is a one-to-one correspondence between the set of points $\{C_k | k \in p, \dots, q-1\}$ and the set of integer within the range $0, \dots, \|u(C_{i|j})\|_1 - 1$.

The set of upper (resp. lower) *leaning points* of $C_{i|j}$, denoted by $L^{\uparrow}(C_{i|j})$ (resp. $L^{\downarrow}(C_{i|j})$), are the set of points of $C_{i|j}$ having a remainder equal to $\|u(C_{i|j})\|_1 - 1$ (resp. 0). The set of upper (resp. lower) leaning points of fig. 1 have a remainder equal to 6 (resp. 0).

The use of the terms "upper" and "lower" really becomes evident when $C_{i|j}$ is in the first octant, *i.e.* $0 \leq y(u(C_{i|j})) < x(u(C_{i|j}))$. Note that l, as introduced in definition 1, can be the index of any lower leaning point. For all $\gamma \in \{\uparrow, \downarrow\}$, the first and last leaning points are the one having respectively the smallest and greatest index, *i.e.* $L^{\gamma}_{min}(C_{i|j}) = \arg\min_{C_k \in L^{\gamma}(C_{i|j})} k$ and $L^{\gamma}_{max}(C_{i|j}) = \arg\max_{C_k \in L^{\gamma}(C_{i|j})} k$.

Although the equation $det(\vec{u}, \vec{v}) = r$ has infinitely many solutions when $r \in \mathbb{Z}$, it has sometimes exactly one solution, for instance when $0 \leq y(\vec{u}) < x(\vec{u})$ and $0 \leq x(\vec{v}) < x(\vec{u})$.

Definition 2 (Bezout vectors) *Let $C_{i|j}$ be a DSS assumed to be without loss of generality in the first octant,* i.e. $0 \leq y(u(C_{i|j})) < x(u(C_{i|j}))$. *The* Bezout vector *of $C_{i|j}$, denoted by $v^+(C_{i|j})$ (resp. $v^-(C_{i|j})$) is such that $det(u(C_{i|j}), v^+(C_{i|j})) = 1$ (resp. $det(u(C_{i|j}), v^-(C_{i|j})) = -1$) and $x(v^+(C_{i|j})) \leq x(u(C_{i|j}))$ (resp. $x(v^-(C_{i|j})) \leq x(u(C_{i|j}))$).*

The *slope* of a vector $\vec{u}$ is the function $\rho : \mathbb{Z}^2 \to \mathbb{Q} \cup \infty$ defined by $\rho(\vec{u}) = \frac{y(\vec{u})}{x(\vec{u})}$. The slope of a DSS S is the slope of its direction vector $u(S)$.

The order $>$ on slopes is such that $\rho(\vec{u}) > \rho(\vec{v})$ when $det(\vec{u}, \vec{v}) < 0$. In the sequel, we will take profit of the fact that comparing slopes and computing remainders both imply a determinant calculus. For instance, we will use the following property:

Property 1 *Let S be a DSS, M and M' be two points of remainder r and r' respectively. The following equality is true: $det(u(S), \overrightarrow{MM'}) = r' - r$.*

Proof

On the one hand, we have $r = r(S, M) = det(u(S), \overrightarrow{L^{\downarrow}_{min}(S)M})$. On the other hand, we have $r' = r(S, M') = det(u(S), \overrightarrow{L^{\downarrow}_{min}(S)M'})$. From basic properties of determinant, the quantity $det(u(S), \overrightarrow{L^{\downarrow}_{min}(S)M'}) - det(u(S), \overrightarrow{L^{\downarrow}_{min}(S)M})$ is equal to $det(u(S), \overrightarrow{L^{\downarrow}_{min}(S)M'} - \overrightarrow{L^{\downarrow}_{min}(S)M})$, which is obviously equal to $det(u(S), \overrightarrow{MM'})$. Therefore $r' - r = det(u(S), \overrightarrow{MM'})$. ∎

Property 1 is illustrated in fig. 2.

In the general case, the order $>$ on slopes is not transitive, *i.e* $\rho(\vec{u}) > \rho(\vec{v})$ and $\rho(\vec{v}) > \rho(\vec{w})$ does not imply that $\rho(\vec{u}) > \rho(\vec{w})$. However, the order is sometimes transitive, for instance when $x(\vec{u})$, $x(\vec{v})$ and $x(\vec{w})$ are positive. Indeed, $\rho(\vec{u}) > \rho(\vec{v})$ implies that $(x(\vec{u})y(\vec{v}) - y(\vec{u})x(\vec{v})) > 0$. Since $x(\vec{u})$, $x(\vec{v})$ are positive, this is equivalent to $\frac{y(\vec{v})}{x(\vec{v})} > \frac{y(\vec{u})}{x(\vec{u})}$. Similarly $\rho(\vec{v}) > \rho(\vec{w})$ implies that $(x(\vec{v})y(\vec{w}) - y(\vec{v})x(\vec{w})) > 0$. Since $x(\vec{v})$, $x(\vec{w})$ are positive, this is equivalent to $\frac{y(\vec{w})}{x(\vec{w})} > \frac{y(\vec{v})}{x(\vec{v})}$. Therefore, $\frac{y(\vec{w})}{x(\vec{w})} > \frac{y(\vec{v})}{x(\vec{v})} > \frac{y(\vec{u})}{x(\vec{u})}$. The inequality $\frac{y(\vec{w})}{x(\vec{w})} > \frac{y(\vec{u})}{x(\vec{u})}$ is equivalent to $(x(\vec{u})y(\vec{w}) - y(\vec{u})x(\vec{w})) > 0$ and thus $\rho(\vec{u}) > \rho(\vec{w})$.

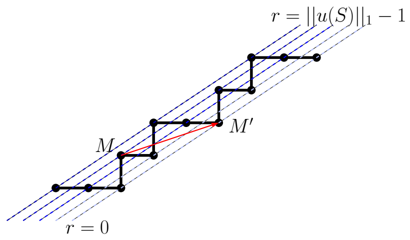
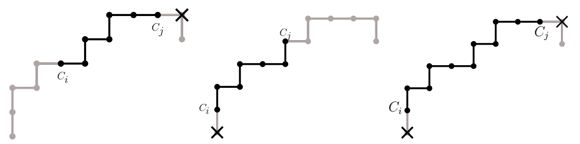


Figure 2: The dotted lines are of directional vector $u(S)$ and of remainder $0 \leq r \leq \|u(S)\|_1 - 1$. If two points M and M' are of respective remainders r and r' such that $r > r'$, then $det(u(S), \overrightarrow{MM'}) < 0$, which means that the slope of $u(S)$ is greater than the slope of $\overrightarrow{MM'}$.

2.2 Cover of MSs

A maximal DSS or segment (MS for short) is a DSS $C_{i|j}$ that cannot be extended neither at the front nor at the back. It was first introduced as a digital tangent in [23] in order to extend the symmetric digital tangent of Vialard [24].

Definition 3 (MS) *A DSS $C_{i|j}$ that cannot be extended at the front,* i.e. $j = n$ *or* $C_{i|j+1}$ *is not a DSS, is said maximal at the front and is denoted by $S_{i\rightarrow}$. Similarly, a DSS $C_{i|j}$ that cannot be extended at the back,* i.e. $i = 1$ *or* $C_{i-1|j}$ *is not a DSS, is said maximal at the back and is denoted by $S_{\leftarrow j}$. Finally, a DSS $C_{i|j}$ that is both maximal at the front and maximal at the back,* i.e. $C_{i|j} = S_{i\rightarrow} = S_{\leftarrow j}$, *is a maximal segment (fig. 3).*



(a) (b) (c)

Figure 3: (a) A DSS $S_{i\rightarrow}$ maximal at the front, (b) a DSS $S_{\leftarrow j}$ maximal at the back, and (c) a MS.

The cover of a digital curve (or tangential cover [14]) was first introduced as a set of digital tangents in [23]. It contains all DSS segmentations, one of which has the minimal number of DSS [25]. It has been used for estimations of length [26], tangents [23, 27, 18] or curvature [23, 28, 29] as well as for convex and concave parts decomposition [14, 13, 15].

Definition 4 (Cover) *The* cover *of a digital curve $C_{i|j}$ is the whole set of maximal segments contained in $C_{i|j}$.*

By definition, the cover exists and is unique for any digital curve. In the sequel, the MSs of the cover of $C_{i|j}$ are ordered by the index of their first point, *i.e.*, they are considered as the sequence $S_{k_1\rightarrow}, S_{k_2\rightarrow}, \ldots, S_{k_M\rightarrow}$ where $i = k_1 < k_2 < \ldots < k_M < j$ and $S_{k_M\rightarrow} = C_{k_M|j}$. Since the cover contains all existing MSs, for all $m \in 1, \ldots, M-1$, for all $k \in k_m, \ldots, k_{m+1}-1$, $S_{k\rightarrow} \subset S_{k_m\rightarrow}$ and $S_{k\rightarrow}$ is not a MS.

There exists an elegant algorithm that computes the cover of a digital curve in linear time [23, 18]. An illustration of the output of this algorithm is depicted in fig. 4.

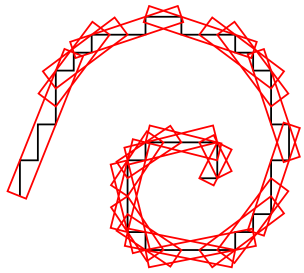


Figure 4: Cover of a digital curve. Each MS is depicted with a red bounding box.

The mechanism can be coarsely described as follows: given a MS, the next one is computed first by removing points from the back of the segment until it is no longer maximal at the front and then by adding points at the front of the segment until it is maximal at the front.

The key tasks are adding [30] and removing [18] a point at one extremity of a DSS in constant time and space. The recognition algorithm of Debled and Reveillès [30] provides a way to test the maximality of a DSS. Given a DSS $C_{k|l}$, the algorithm decides whether $C_{k|l+1}$ is a DSS too or not. Moreover, if $C_{k|l+1}$ is a DSS, the direction vector, the Bezout vectors, the first and last upper and lower leaning points of $C_{k|l+1}$ can be computed from those of $C_{k|l}$ in constant time [30, 31].

Table 1 sums up these results with simplified notations. Row 0 contains the possible values (columns B to E) of the remainder of C_{l+1} with respect to the DSS $C_{k|l}$ (column A). Rows 1 to 7 contain the values (columns B-D), assigned to the parameters of $C_{k|l+1}$ (column A). For instance, if $r(C_{k|l}, C_{l+1}) = -1$, then $u(C_{k|l+1}) = \overrightarrow{L^{\downarrow}_{min}(C_{k|l})C_{l+1}}$ (column D, row 1).

The algorithm of Debled and Reveillès [30] applied on simply 4-connected digital curve is noth-

Table 1: If the remainder of C_{l+1} with respect to the DSS $C_{k|l}$ is greater than -1 and smaller than $\|u(C_{k|l})\|_1$, $C_{k|l+1}$ is a DSS. The parameters of $C_{k|l+1}$ (column A) are computed from the parameters of $C_{k|l}$ (columns B-D) [30, 31].

	A	B	C	D	E
0	r	$= \|u\|_1$	$[0, \|u\|_1[$	$= -1$	else
1	u	$\overrightarrow{L^{\uparrow}_{min}C_{l+1}}$	still a DSS, no change	$\overrightarrow{L^{\downarrow}_{min}C_{l+1}}$	not a DSS anymore
2	v^-	u		$\overrightarrow{L^{\downarrow}_{min}C_{l+1}} - u$	
3	v^+	$\overrightarrow{L^{\uparrow}_{min}C_{l+1}} - u$		u	
4	$L^{\downarrow}_{min}$	$L^{\downarrow}_{max}$		$L^{\downarrow}_{min}$	
5	$L^{\downarrow}_{max}$	$L^{\downarrow}_{max}$		C_{l+1}	
6	$L^{\uparrow}_{min}$	$L^{\uparrow}_{min}$		$L^{\uparrow}_{max}$	
7	$L^{\uparrow}_{max}$	C_{l+1}		$L^{\uparrow}_{max}$	

ing else than the algorithm of Kovalevsky [32] under the arithmetic formalism of Reveillès [20].

Obviously, if the point C_k is removed from the back of the DSS $C_{k|l}$, $C_{k+1|l}$ is a DSS too, but possibly with a different direction vector. Thinking as if C_k was added to $C_{k+1|l}$ leads to a reversed algorithm that updates the leaning points, the direction and Bezout vectors of $C_{k+1|l}$ from those of $C_{k|l}$ [18].

2.3 Intersection of two consecutive MSs

The intersection of two consecutive MSs is never empty, and contains at least two points. The proof, based on the fact that any three consecutive points of a digital curve (assumed to be simply connected) form a DSS, is left to the reader.

We will show below that the remainder of the points that bound the intersection between two consecutive MSs is strongly constrained. The following lemma has already been shown in [31, lemma 2] and will be completed in corollary 1.

Lemma 1 *Let $C_{k|l}$ and $C_{k'|l'}$ be two consecutive MSs of the cover of a digital curve $C_{i|j}$ and their intersection $C_{k'|l}$.*

The remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1 or $\|u(C_{k'|l})\|_1$.

Proof

The remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are neither strictly greater than $\|C_{k'|l}\|_1$ nor strictly smaller than -1, because $C_{k'-1|l}$ and $C_{k'|l+1}$ are DSSs.

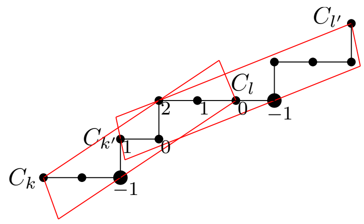


Figure 5: Illustration of Lemma 1 : $C_{k'|l}$ is a DSS of slope $\frac{1}{2}$, the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1.

Let us assume that the remainder of $C_{k'-1}$ with respect to $C_{k'|l}$ belongs to the range $0, \dots, \|u(C_{k'|l})\|_1 - 1$. Whatever the remainder of C_{l+1} with respect to $C_{k'|l}$ in the range $-1, \dots, \|u(C_{k'|l})\|_1$, $C_{k'-1|l+1}$ would be a DSS (table 1, row 0, columns B and D), which raises a contradiction. A similar result is obtained if we swap the remainder of $C_{k'-1}$ and the one of C_{l+1}. Therefore, the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are either equal to $\|u(C_{k'|l})\|_1$ or -1.

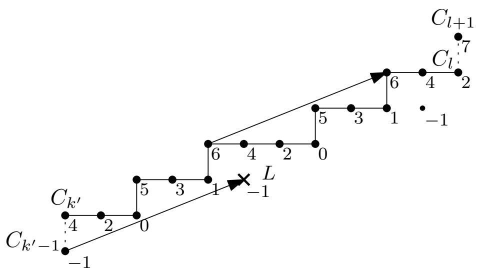
Let us now assume that the remainder of $C_{k'-1}$ with respect to $C_{k'|l}$ equals to -1, whereas the remainder of C_{l+1} with respect to $C_{k'|l}$ equals to $\|u(C_{k'|l})\|_1$. In addition, let us assume without loss of generality that $C_{k'|l}$ is in the first octant, *i.e.* $0 \le y(u(C_{k'|l})) < x(u(C_{k'|l}))$ (fig. 6.a).

Table 1 says that $C_{k'|l+1}$ is a DSS of slope greater than the one of $C_{k'|l}$. More precisely $det(u(C_{k'|l}), u(C_{k'|l+1}))$ is equal to $det(v^+(C_{k'|l+1}), u(C_{k'|l+1}))$ (table 1, column D, row 3), which is equal to 1 according to definition 2.

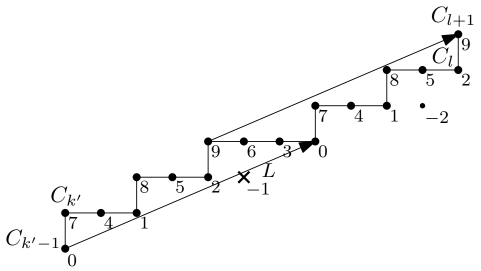
The first upper leaning point of $C_{k'|l}$ and $C_{k'|l+1}$ are confounded but have remainders respectively equal to $\|u(C_{k'|l})\|_1 - 1$ and $\|u(C_{k'|l+1})\|_1 - 1$. Let us denote by L their translation by the vector $\vec{s}$ $(1, -1)$ (fig. 6.a). Due to property 1, the remainder of L with respect to $C_{k'|l}$ is equal to $r(C_{k'|l}, L^{\uparrow}_{min}(C_{k'|l})) + det(u(C_{k'|l}), \vec{s})$. Since $det(u(C_{k'|l}), \vec{s}) = -\|u(C_{k'|l})\|_1$, then $r(C_{k'|l}, L) = -1$. We can similarly show that $r(C_{k'|l+1}, L) = -1$.

In addition, due to the definition of L and the remainders of L and $C_{k'-1}$ with respect to $C_{k'|l}$, the vector $\overrightarrow{C_{k'-1}L}$ is equal to $u(C_{k'|l})$ (fig. 6.a). As a consequence, due to property 1, the remainder of $C_{k'-1}$ with respect to $C_{k'|l+1}$ is equal to $r(C_{k'|l+1}, L) + det(u(C_{k'|l}), u(C_{k'|l+1}))$, which is equal to $-1 + 1 = 0$. $C_{k'-1|l+1}$ is thus a DSS in that case (fig. 6.b), which raises a contradiction.

Once again, a similar result is obtained if we swap the remainder of $C_{k'-1}$ and the one of C_{l+1}. Therefore, the remainders of $C_{k'-1}$ and C_{l+1} with respect



(a)



(b)

Figure 6: Illustration of the proof of Lemma 1. (a) The remainder of $C_{k'-1}$ with respect to $C_{k'|l}$ equals to -1, whereas the remainder of C_{l+1} with respect to $C_{k'|l}$ equals to $\|u(C_{k'|l})\|_1 = 7$. (b) $C_{k'-1|l+1}$ is a DSS, which means that $C_{k'|l}$ cannot be the common part of two consecutive MSs.

to $C_{k'|l}$ are both equal to $\|u(C_{k'|l})\|_1$ or -1.

We show in the following lemma that the slope of the common part of two consecutive MSs is bounded by the slopes of all the segments that contain this part. A previous result about the intersection of two digital straight lines [33, theorem 5] is quite similar.

Lemma 2 *Let $C_{k|l}$ and $C_{k'|l'}$ be two consecutive MSs of the cover of a digital curve $C_{i|j}$. The slope of their intersection $C_{k'|l}$ is bounded by the slope of the DSSs maximal at the front or at the back that contain $C_{k'|l}$,* i.e. *for all $k^\circ \in k, \dots, k'-1$ and for all $l^\circ \in l+1, \dots, l'$, $\rho(u(C_{k^\circ|l})) > \rho(u(C_{k'|l})) > \rho(u(C_{k'|l^\circ}))$ if the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1 and $\rho(u(C_{k^\circ|l})) < \rho(u(C_{k'|l})) < \rho(u(C_{k'|l^\circ}))$ if they are both equal to $\|u(C_{k'|l})\|_1$.*

Proof

Without loss of generality, let us assume that $C_{k'|l}$ is in the first octant. Since the coordinates of the points of a DSS both monotonously increase or decrease and since $C_{k'|l}$ has at least two points, our hypothesis implies that all the points from C_k to C_l are sorted along the x-coordinate such that $x(C_k) \leq x(C_{k+1}) \leq \dots \leq x(C_{l'-1}) \leq x(C_{l'})$.

Due to lemma 1, the remainder of C_{l+1} with respect to $C_{k'|l}$ is either equal to -1 or $\|C_{k'|l}\|_1$. We will focus on the first case and prove by induction that for all $l^\circ \in l+1, \dots, l'$, $\rho(u(C_{k'|l})) > \rho(u(C_{k'|l^\circ}))$. In the second case, we can similarly show that for all $l^\circ \in l+1, \dots, l'$, $\rho(u(C_{k'|l})) < \rho(u(C_{k'|l^\circ}))$.

The proof is based on the three following arguments, which are proved hereafter :

1. $\rho(v^+(C_{i|j})) > \rho(u(C_{i|j}))$ for all DSS $C_{i|j}$
2. $\rho(u(C_{k'|l})) = \rho(v^+(C_{k'|l+1}))$
3. $\rho(v^+(C_{k'|l^\circ})) \geq \rho(v^+(C_{k'|l^\circ+1}))$ for all l° in $l, \dots l'-1$.

From 2) and 3) we deduce $\rho(u(C_{k'|l})) \geq \rho(v^+(C_{k'|l^\circ}))$ for all l° in $l+1, \dots l'$. With argument 1) we conclude that $\rho(u(C_{k'|l})) > \rho(u(C_{k'|l^\circ}))$ for all l° in $l+1, \dots l'$.

Proof of 1) - direct from the definition of v^+ (definition 2).

Proof of 2) - Since the remainder of C_{l+1} with respect to $C_{k'|l}$ is assumed to be equal to -1, the result is direct from table 1, column D, row 3. In fig. 7, the slope of $C_{k'|l+1}$ is equal to $\frac{2}{5}$, whereas the slope of $C_{k'|l}$ is equal to $\frac{1}{2}$.

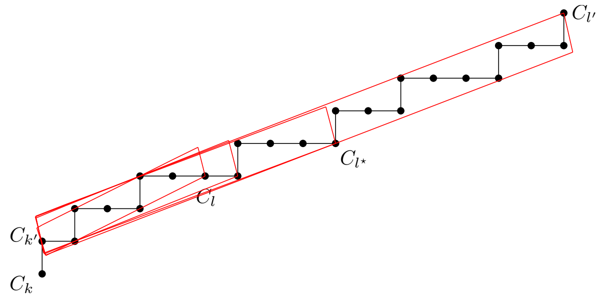


Figure 7: Illustration of the proof of Lemma 2. The slope of the common part $C_{k'|l}$ is equal to $\frac{1}{2} = 0.5$. Some of the DSSs maximal at $C_{k'}$ are depicted with a red bounding box: $C_{k'|l+1}$, whose slope is equal to $\frac{2}{5} = 0.4$, illustrates argument 2), $C_{k'|l^\star}$, whose slope is equal to $\frac{3}{8} = 0.375$, illustrates case (ii) of argument 3) and $C_{k'|l'}$, whose slope is equal to $\frac{5}{13} \approx 0.385$, illustrates case (iii) of argument 3). All the DSSs maximal at the back contained in the MS $C_{k'|l'}$ have a slope lower than the one of the common part $C_{k'|l}$.

Proof of 3) - Three cases can occur, according to the value of $r(C_{k'|l^\circ}, C_{l^\circ+1})$.

(i) If $r(C_{k'|l^\circ}, C_{l^\circ+1})$ belongs to the range $0, \ldots, \|u(C_{k'|l^\circ})\|_1 - 1$, we have $v^+(C_{k'|l^\circ}) = v^+(C_{k'|l^\circ+1})$.

(ii) If $r(C_{k'|l^\circ}, C_{l^\circ+1}) = -1$, from table 1, column D, row 3, we have $v^+(C_{k'|l^\circ+1}) = u(C_{k'|l^\circ})$, and conclude that $\rho(v^+(C_{k'|l^\circ})) > \rho(v^+(C_{k'|l^\circ+1}))$ due to argument 1). For instance, in fig. 7, the slope of $C_{k'|l^\star-1}$ is equal to $\frac{2}{5} = 0.4$, which is greater than the slope of $C_{k'|l^\star}$ that is equal to $\frac{3}{8} = 0.375$.

(iii) If $r(C_{k'|l^\circ}, C_{l^\circ+1}) = \|u(C_{k'|l^\circ})\|_1$, the slope of $C_{k'|l^\circ}$ is strictly lower than the one of $C_{k'|l^\circ+1}$. For instance, in fig. 7, the slope of $C_{k'|l'-1}$ is equal to $\frac{3}{8} = 0.375$, which is lower than the slope of $C_{k'|l'}$ that is equal to $\frac{5}{13} \approx 0.385$. But we can still conclude.

Let us denote by q the positive integer such that $q.u(C_{k|l^\circ}) = \overrightarrow{L^{\uparrow}_{min}(C_{k|l^\circ})L^{\uparrow}_{max}(C_{k|l^\circ})}$. From table 1, column B, $u(C_{k|l^\circ+1})$ is equal to $\overrightarrow{L^{\uparrow}_{min}(C_{k|l^\circ})C_{l^\circ+1}}$, which is equal to $q.u(C_{k|l^\circ}) + v^+(C_{k|l^\circ})$. From table 1, column B, row 3, we have $v^+(C_{k|l^\circ+1}) = (q-1).u(C_{k|l^\circ}) + v^+(C_{k|l^\circ})$.

As a consequence, $det(v^+(C_{k|l^\circ}), v^+(C_{k|l^\circ+1}))$ is equal to $det(v^+(C_{k|l^\circ}), (q-1).u(C_{k|l^\circ}) + v^+(C_{k|l^\circ}))$, which is equal to $-q+1$. We can thus conclude that $det(v^+(C_{k|l^\circ}), v^+(C_{k|l^\circ+1})) \leq 0$ and that $\rho(v^+(C_{k'|l^\circ})) \geq \rho(v^+(C_{k'|l^\circ+1}))$.
Due to lemma 1, since the remainder of C_{l+1} with respect to $C_{k'|l}$ is assumed to be equal to -1, the remainder of $C_{k'-1}$ is equal to -1 too.

We can similarly show that for all $k^\circ \in k, \ldots, k'-1$, $\rho(u(C_{k^\circ|l})) > \rho(u(C_{k'|l}))$, which concludes the proof. $\blacksquare$

The results of the two previous lemmas lead to the following crucial corollary:

Corollary 1 *Let $C_{k|l}$ and $C_{k'|l'}$ be two consecutive MSs of the cover of a digital curve $C_{i|j}$. We have $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ (resp. $\rho(u(C_{k|l})) < \rho(u(C_{k'|l'}))$) if and only if the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1 (resp. $\|u(C_{k'|l})\|_1$).*

Proof

$\Rightarrow$ Due to lemma 2, $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ (resp. $\rho(u(C_{k|l})) < \rho(u(C_{k'|l'}))$) if the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1 (resp. $\|u(C_{k'|l})\|_1$).

$\Leftarrow$ Due to lemma 1, the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both either equal to -1 or $\|u(C_{k'|l})\|_1$. If $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ (resp. $\rho(u(C_{k|l})) < \rho(u(C_{k|l}))$), the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ cannot be equal to $\|C_{k'|l}\|_1$ (resp. -1), because if they were, due to lemma 2, $\rho(u(C_{k|l})) < \rho(u(C_{k'|l'}))$ (resp. $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$), which raises a contradiction. As a consequence, the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1 (resp. $\|C_{k'|l}\|_1$) if $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ (resp. $\rho(u(C_{k|l})) < \rho(u(C_{k'|l'}))$). $\blacksquare$

3 Decomposition into convex and concave parts

In this section, we propose a definition of convex and concave parts related to the slope of the MSs of the cover of a digital curve. Previous results, especially corollary 1, lead to interesting local convexity properties that provide a simple, online and linear-time algorithm of decomposition.

3.1 Maximal convex and concave parts

Many definitions of digital convexity exist. The first one, based on digitization, comes from Sklansky [3]. This definition is not convenient because unconnected sets may be considered as digitally convex. Later, Kim [4, 5], Kim and Rosenfeld [6, 7], Kim and Sklansky [8], Chassery [9] and Ronse [10] proposed other definitions of digital convexity. Most of these definitions may be proved to be equivalent for simply connected sets [5, 4, 10, 11].

However, as in [13, 14, 15], we define convex and concave parts by the means of the slope of MSs.

Definition 5 (Convex and concave parts) *A digital curve $C_{i|j}$ is convex (resp. concave) if and only if the slope of any two consecutive MSs of its cover decreases (resp. increases),* i.e. *for all $m \in 1, \ldots, M-1$, $S_{k_m \rightarrow}$ and $S_{k_{m+1} \rightarrow}$ are two consecutive MSs such that $\rho(u(S_{k_m \rightarrow})) > \rho(u(S_{k_{m+1} \rightarrow}))$ (resp. $\rho(u(S_{k_m \rightarrow})) < \rho(u(S_{k_{m+1} \rightarrow}))$).*

This definition is valid for open digital curves, contrary to most of previous definitions, which are only valid for digital sets. For instance, the spiral-shaped digital curve of fig. 4 is convex because the slope of any two consecutive MSs of its cover decreases.

Note that a convex digital curve $C_{i|j}$ is concave if the points are scanned from j to i and conversely.

Moreover, a digital curve that is neither convex nor concave may be straightforwardly decomposed into maximal convex and concave parts according to the slope of the MSs of its cover.

Definition 6 (Maximal convex and concave parts) *Any maximal sequence of consecutive MSs of decreasing (resp. increasing) slope determines a maximal convex (resp. concave) part.*

Since the cover is unique, such a decomposition into maximal convex and concave parts is unique too (fig. 8). Moreover, one MS may belong to one maximal part, either convex or concave, respectively in blue and green in fig. 8, or may belong to both a maximal convex part and a maximal concave part, in yellow in fig. 8.

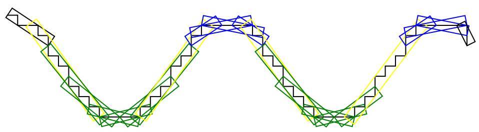


Figure 8: Decomposition of a digital curve into maximal convex and concave parts. A MS is depicted by a bounding box: blue if the MS is in a convex part, green in a concave part, yellow if it makes the transition between a convex and a concave part.

Note that polygons have similar properties. Their maximal convex and concave parts are maximal sequences of edges of respectively decreasing and increasing slope. One edge may belong to both a convex and a concave part. Our definitions are thus quite natural.

3.2 Local criterion for checking convexity

Once the cover is computed, decomposing a digital curve into maximal convex and concave parts is trivial. However, further results, relying on corollary 1, are required to derive an online algorithm.

Theorem 1 *If a digital curve $C_{i|j}$ is convex (resp. concave), then any subpart $C_{i'|j'}$ of $C_{i|j}$ is convex (resp. concave) too.*

Proof

We will assume that $C_{i|j}$ is convex but a similar proof can be derived in the case where $C_{i|j}$ is assumed to be concave. First, we will prove by induction that $C_{i|j'}$ is convex for all $j' \in i+1, \dots, j$.

Let the property $P_{j'}$ be "$C_{i|j'}$ is convex" for all $j' \in i+1, \dots, j$. P_j is obviously true because $C_{i|j}$ is convex. Now, we will show that $P_{j'}$ is true when $P_{j'+1}$ is assumed to be true for some $j' \in i+1, \dots, j-1$.

Let C_{k_m} and $C_{k_{m+1}}$ be the first points of the last two MSs $S_{k_m \rightarrow}$ and $S_{k_{m+1} \rightarrow} = C_{k_{m+1}|j'+1}$ of the cover of $C_{i|j'+1}$. Moreover, let C_{l_m} be the last point of $S_{k_m \rightarrow}$ so that $S_{k_m \rightarrow} = C_{k_m|l_m}$. Fig. 9 illustrates the notations used in this proof.

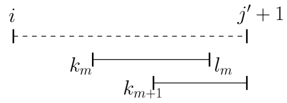


Figure 9: Illustration of the indices used in the proof of Theorem 1.

Two different cases occur:

(i) if $j'+1 = l_m + 1$, then $j' = l_m$ and the cover $C_{i|j'}$ is made up of the m first MSs of the cover of $C_{i|j'+1}$ but does not include the last one, *i.e.* $S_{k_{m+1} \rightarrow}$. Due to the induction hypothesis, $C_{i|j'+1}$ is convex and the slope of any two consecutive MSs of its cover decreases. As a consequence, $C_{i|j'}$ is convex too.

(ii) otherwise, $j'+1 > l_m + 1$, which implies $j' > l_m$ and the cover of $C_{i|j'}$ is made up with the m first MSs of the cover of $C_{i|j'+1}$, plus the segment $C_{k_{m+1}|j'}$, which is contained in $S_{k_{m+1} \rightarrow} = C_{k_{m+1}|j'+1}$. Due to the induction hypothesis, $C_{i|j'+1}$ is convex and for all $\mu \in 1, \dots, m-1$, $\rho(u(S_{k_\mu \rightarrow})) > \rho(u(S_{k_{\mu+1} \rightarrow}))$. It remains to prove that $\rho(u(S_{k_m \rightarrow})) \geq \rho(u(C_{k_{m+1}|j'}))$.

Due to corollary 1, $\rho(u(S_{k_m \rightarrow})) > \rho(u(S_{k_{m+1} \rightarrow}))$ implies that the remainder of C_{l_m+1} with respect to $C_{k_{m+1}|l_m}$ is equal to -1. And conversely, if the remainder of C_{l_m+1} with respect to $C_{k_{m+1}|l_m}$ is equal to -1, $\rho(u(S_{k_m \rightarrow})) > \rho(u(C_{k_{m+1}|j'}))$. Thus, $C_{i|j'}$ is convex too in this case.

We have shown by induction that for all $j' \in i+1, \dots, j$, $C_{i|j'}$ is convex. Similarly, we can show that for all $i' \in i, \dots, j-1$, $C_{i'|j}$ is convex. Therefore, $C_{i'|j'}$ is convex, which concludes the proof. ∎

Due to theorem 1, an incremental decomposition of a digital curve into convex and concave parts makes sens. In order to decide if a given MS $C_{k|l}$ is the last one of a maximal convex (resp. concave) part, we have to check if its slope is smaller (resp. greater) than the slope of the next MS $C_{k'|l'}$ according to definition 5.

Though, we show in the following theorem that it is enough to compute the remainder of C_{l+1} with respect to $C_{k|l}$ without considering the next MS $C_{k'|l'}$:

Theorem 2 *Let $C_{k|l}$ and $C_{k'|l'}$ be two consecutive MSs of the cover of a digital curve $C_{i|j}$. The following equivalences are true: $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'})) \Leftrightarrow r(C_{k|l}, C_{l+1}) < -1$ and $\rho(u(C_{k|l})) < \rho(u(C_{k'|l'})) \Leftrightarrow r(C_{k|l}, C_{l+1}) > \|u(C_{k|l})\|_1$.*

Proof

We will only show that the first equivalence is true (fig. 10.a), proving the second one requires a similar proof.

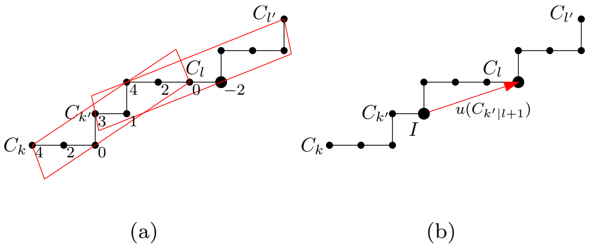


Figure 10: Illustration of theorem 2 (a) and its proof (b). The parts $C_{k|l}$ and $C_{k'|l'}$ are two consecutive MSs of decreasing slopes and the remainder of C_{l+1} with respect to $C_{k|l}$ is strictly lower than -1 (-2 here). The point I is the translated of C_{l+1} by the vector $-u(C_{k'|l+1})$. It belongs to $C_{k|l}$ and the value of its remainder is used to bound $r(C_{k|l}, C_{l+1})$, knowing that $\rho(u(C_{k|l})) > \rho(u(C_{k'|l+1}))$.

Since $C_{k|l}$ is a MS, $C_{k|l+1}$ is not a DSS. In other words, $r(C_{k|l}, C_{l+1}) > \|u(C_{k|l})\|_1$ or $r(C_{k|l}, C_{l+1}) < -1$. Therefore, to conclude, it is enough to prove that $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'})) \Leftrightarrow r(C_{k|l}, C_{l+1}) < \|u(C_{k|l})\|_1$.

We first prove that $\rho(u(C_{k|l})) > \rho(u(C_{k'|l+1})) \Leftrightarrow r(C_{k|l}, C_{l+1}) < \|u(C_{k|l})\|_1$.

Let us consider the DSS $C_{k'|l+1}$. By definition, $C_{k'|l+1}$ contains at least two points P and Q such that $Q = P + u(C_{k'|l+1})$. Since C_{l+1} is the last point of $C_{k'|l+1}$, the point I, image of C_{l+1} after a translation by the vector $-u(C_{k'|l+1})$, necessarily belongs to $C_{k'|l}$. Since $C_{k'|l}$ is included in $C_{k|l}$, I belongs to $C_{k|l}$ too (fig. 10.b).

Due to property 1, $r(C_{k|l}, C_{l+1}) = r(C_{k|l}, I) + det(u(C_{k|l}), u(C_{k'|l+1}))$. Since I belongs to $C_{k|l}$, the remainder of I with respect to $C_{k|l}$ is greater than 0 and strictly smaller than $\|u(C_{k|l})\|_1$. Thus, $det(u(C_{k|l}), u(C_{k'|l+1})) < 0 \Leftrightarrow r(C_{k|l}, C_{l+1}) < \|u(C_{k|l})\|_1$. However $det(u(C_{k|l}), u(C_{k'|l+1})) < 0$ is equivalent to $\rho(u(C_{k|l})) > \rho(u(C_{k'|l+1}))$.

It remains to show that $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'})) \Leftrightarrow \rho(u(C_{k|l})) > \rho(u(C_{k'|l+1}))$.

Due to corollary 1, $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ as well as $\rho(u(C_{k|l})) > \rho(u(C_{k'|l+1}))$ imply that the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are both equal to -1. Conversely, if the remainders of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ are equal to -1, $\rho(u(C_{k|l})) > \rho(u(C_{k'|l'}))$ and $\rho(u(C_{k|l})) > \rho(u(C_{k'|l+1}))$, which concludes the proof. ∎

Similar results may be found in [17][Theorem 9] and [18, lemma 1] but theorem 2 gives a fuller answer to Eckhardt's question [16]: how far one can decide whether a part of a digital curve is convex or not by a method that is *as local as possible*? Our answer is that the smallest digital pattern required for checking convexity is given by a MS, *plus* at least one of the two points located just before and after this segment. Within a smaller pattern, which would be a MS, all parts of any digital curves would be considered straight and thus both convex and concave, which brings nothing.

Looking at the remainders of the two points that bound a MS S is a way of classifying the local configuration of the digital curve:

- if the remainders of the two points are strictly less than -1 (resp. strictly greater than $\|u(S)\|_1$), S belongs to a convex (resp. concave) part (fig. 11.a and b).
- if the remainder of one of the two points is strictly less than -1 and the remainder of the other is strictly greater than $\|u(S)\|_1$, S makes the transition between a convex and a concave part (fig. 11.c).

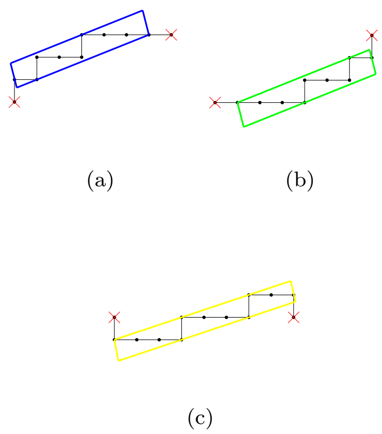


Figure 11: MS in a convex part in (a), in a concave part in (b), in an inflection part in (c).

3.3 Algorithm

Theorem 2 straightforwardly leads to algorithm 1. The maximal convex and concave parts of an open digital curve are retrieved in the course of the cover computation. A MS $C_{k|l-1}$ that belongs to a convex (resp. concave) part and such that the remainder of C_l with respect to $C_{k|l-1}$ is greater than $\|u(C_{k|l-1})\|_1$ (resp. strictly smaller than -1) is the end of a maximal convex (resp. concave) part and the beginning of a concave (resp. convex) part (lines 16-20). Fig. 8 illustrates the output of algorithm 1.

Algorithm 1: Convex and concave parts decomposition

Input: a digital curve $C_{i|j}$
Output: The list $\mathcal{L}$ of the maximal convex and concave parts of $C_{i|j}$

1 $\mathcal{L} \leftarrow \emptyset$; `/* list of convex and concave parts */`
2 $k' \leftarrow i$; `/* beginning of the current part */`
3 $k \leftarrow k'$; `/* beginning of the current segment */`
4 $l \leftarrow i+1$; `/* index of the current point */`
5 $isConvex$; `/* true if convex, false otherwise */`
`/* initialization */`
6 **while** $l \leq j$ *and* $C_{k|l}$ *is a DSS* **do**
7     $l \leftarrow l+1$; `/* add a point to the front */`
8 **if** $l \leq j$ **then**
9   **if** $r(C_{k|l-1}, C_l) < -1$ **then** $isConvex \leftarrow true$;
10   **else** $isConvex \leftarrow false$;
`/* body */`
11 **while** $l \leq j$ **do**
12   **while** $C_{k|l}$ *is not a DSS* **do**
13         $k \leftarrow k+1$; `/* remove a point from the back */`
14   **while** $l \leq j$ *and* $C_{k|l}$ *is a DSS* **do**
15         $l \leftarrow l+1$; `/* add a point to the front */`
16   **if** $l \leq j$ **then**
17   **if** *(*$isConvex = true$ *and* $r(C_{k|l-1}, C_l) \geq \|u(C_{k|l-1})\|_1$*) or (*$isConvex = false$ *and* $r(C_{k|l-1}, C_l) < -1$*)* **then**
18             $\mathcal{L} \leftarrow \mathcal{L} + C_{k'|l-1}$; `/* new part stored */`
19             $k' \leftarrow k$; `/* convex ⇔ concave */`
20             $isConvex \leftarrow \neg isConvex$;
21 **return** $\mathcal{L} + C_{k'|l-1}$;

4 Decomposition into digital edges

In this section, we introduce the notions of digital edge, upper and lower hulls and show that any convex (resp. concave) part has a unique upper (resp. lower) hull that can be computed in linear-time. This work has several useful applications in the field of polygonal representation (section 5), which is the motivation of the definitions and results presented below.

4.1 Digital edges, lower and upper hulls

We introduce below the notions of digital edge, upper and lower hulls.

Definition 7 (Digital edge) *A digital curve $C_{i|j}$ is a digital edge if and only if there exists $l \in [i;j]$ such that $\overrightarrow{C_iC_j} = q.\ \vec{u}$, with $q \in \mathbb{Z}^+$ and $\gcd(x(\vec{u}), y(\vec{u})) = 1$, and for all $k \in [i;j]$, $0 \leq det(\vec{u}, \overrightarrow{C_lC_k}) < \|\ \vec{u}\ \|_1$. Moreover, $C_{i|j}$ is a* lower *digital edge if $det(\vec{u}, \overrightarrow{C_iC_l}) = 0$ and a* upper *one otherwise.*

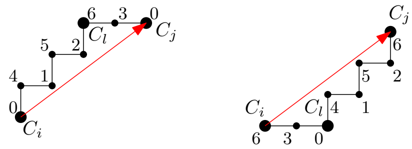


Figure 12: Lower (a) and upper (b) digital edges of slope $\frac{3}{4}$.

By definition, a upper (resp. lower) digital edge is a DSS such that its first and last points are both upper (resp. lower) leaning points. For instance, the DSS depicted in fig. 1 contains a upper digital edge and a lower digital edge whose slopes are the same as its. They are respectively located between the two upper and two lower leaning points of the DSS.

We define now the upper and lower hull of a digital curve.

Definition 8 (upper and lower hulls) *The upper (resp. lower) hull of a digital curve $C_{i|j}$ is a sequence of upper (resp. lower) digital edges $C_{k_0|k_1}, C_{k_1|k_2}, \ldots, C_{k_{M-1}|k_M}$, where $i = k_0 < k_1 < \ldots < k_M = j$, having decreasing (resp. increasing) slope,* i.e. *for all $m \in 1, \ldots, M-1$, $C_{k_{m-1}|k_m}$ and $C_{k_m|k_{m+1}}$ are two consecutive upper (resp. lower) digital edges such that $\rho(u(C_{k_{m-1}|k_m})) > \rho(u(C_{k_m|k_{m+1}}))$ (resp. $\rho(u(C_{k_{m-1}|k_m})) < \rho(u(C_{k_m|k_{m+1}}))$). The ends of each upper (resp. lower) digital edge $C_{k_0}, C_{k_1}, \ldots, C_{k_M}$ are the* vertices *of the upper (resp. lower) hull.*

The upper hull of a digital curve is depicted in fig. 13. Contrary to the cover (fig. 4), any two consecutive DSSs, which are upper digital edges, have only one point in common.

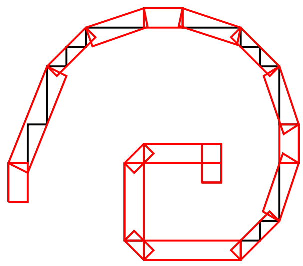


Figure 13: Upper hull of a digital curve. Each upper digital edge is depicted by a red bounding box.

The goal of the following section is to show that a convex digital curve admits a unique upper hull, while a concave digital curve admits a unique lower hull.

4.2 Existence and uniqueness of lower and upper hulls

The following technical lemmas highlight the links between DSSs and digital edges.

We compare in lemma 3, the slope of a DSS S maximal at the front and the one of a digital edge beginning on the last leaning point of S.

Lemma 3 *Let $C_{o|p}$ be a DSS in the first octant,* i.e. $0 \leq y(u(C_{o|p})) < x(u(C_{o|p}))$, *strictly included in the digital curve $C_{i|j}$. Let us assume that there exists a upper (resp. lower) digital edge starting from C_l, the last upper (resp. lower) leaning point of $C_{o|p}$, and ending at C_{l°, with $p < l^\circ \leq j$. If $r(C_{o|p}, C_{p+1}) \leq -1$ (resp. $r(C_{o|p}, C_{p+1}) \geq \|u(C_{o|p})\|_1 - 1$), then $\rho(u(C_{o|p})) > \rho(u(C_{l|l^\circ}))$ (resp. $\rho(u(C_{o|p})) < \rho(u(C_{l|l^\circ}))$).*

Proof

We will only deal with the case where $C_{l|l^\circ}$ is a upper digital edge and $r(C_{o|p}, C_{p+1}) \leq -1$ as illustrated in fig. 14, because the opposite case requires a similar proof.

The first upper leaning point of $C_{l|l^\circ}$, which is assumed to be a digital edge, and the last upper leaning point of $C_{o|p}$ are both located at point C_l. The remainders of C_l with respect to $C_{o|p}$ and $C_{l|l^\circ}$ are respectively equal to $\|u(C_{o|p})\|_1 - 1$ and $\|u(C_{l|l^\circ})\|_1 - 1$. Let us denote by L the translation of C_l by the vector $\vec{s}$ $(1, -1)$. Due to property 1, the remainder of L with respect to $C_{o|p}$ is equal to $r(C_{o|p}, C_l) + det(u(C_{o|p}), \vec{s})$. Since $det(u(C_{o|p}), \vec{s}) = \|u(C_{o|p})\|_1$, then $r(C_{o|p}, L) = -1$. Similarly, $r(C_{l|l^\circ}, L) = -1$.

Due to property 1, $det(u(C_{l|l^\circ}), \overrightarrow{LC_{p+1}}) = r(C_{l|l^\circ}, C_{p+1}) - r(C_{l|l^\circ}, L)$. Since $r(C_{l|l^\circ}, L) = -1$ and $0 \leq r(C_{l|l^\circ}, C_{p+1}) < \|C_{l|l^\circ}\|_1$, $det(u(C_{l|l^\circ}), \overrightarrow{LC_{p+1}}) \geq 1$. Thus, $\rho(u(C_{l|l^\circ})) < \rho(\overrightarrow{LC_{p+1}})$.

Due to property 1, $det(u(C_{o|p}), \overrightarrow{LC_{p+1}}) = r(C_{o|p}, C_{p+1}) - r(C_{o|p}, L)$. Since $r(C_{o|p}, L) = -1$, the sign of $det(u(C_{o|p}), \overrightarrow{LC_{p+1}})$ depends on the value of $r(C_{o|p}, C_{p+1})$.

The remainder of C_{p+1} with respect to $C_{o|p}$ is assumed to be smaller than or equal to -1. Thus, $det(u(C_{o|p}), \overrightarrow{LC_{p+1}}) \leq 0$. In other words, $\rho(u(C_{o|p})) \geq \rho(\overrightarrow{LC_{p+1}})$.

The points of $C_{o|p}$ have increasing x-coordinates, because $C_{o|p}$ is in the first octant. Furthermore, the points of $C_{o|p+1}$ have increasing x-coordinates too, because $C_{i|j}$ is a simple digital curve. Since the intersection between $C_{o|p}$ and $C_{l|l^\circ}$ contains at least two points, $C_{p|p+1}$, the points of $C_{l|l^\circ}$ have also increasing x-coordinates. Therefore, $x(u(C_{o|p}))$, $x(u(C_{l|l^\circ}))$ and $x(\overrightarrow{LC_{p+1}})$ are positive and due to transitivity, $\rho(u(C_{o|p})) \geq \rho(\overrightarrow{LC_{p+1}}) > \rho(u(C_{l|l^\circ}))$ and finally $\rho(u(C_{o|p})) > \rho(u(C_{l|l^\circ}))$. ■

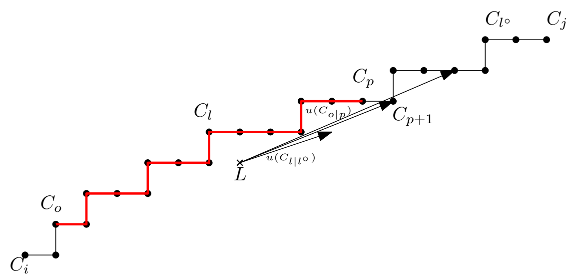


Figure 14: Illustration of the proof of lemma 3. The DSS $C_{o|p}$, which is depicted in red, is maximal at the front because $r(C_{o|p}, C_{p+1}) \leq -1$. Its slope $u(C_{o|p})$ is equal to $\frac{3}{7}$. The slope $u(C_{l|l^\circ})$ of the upper digital edge starting from C_l, the last upper leaning point of $C_{o|p}$, is equal to $\frac{1}{3}$. The slope of $\overrightarrow{LC_{p+1}}$, which is equal to $\frac{2}{5}$, is bounded by $u(C_{l|l^\circ})$ and $u(C_{o|p})$.

Lemma 4 shows that there does not exist any upper (resp. lower) digital edge that shares some of its points with a DSS S and that strictly contains the upper (resp. lower) leaning points of S.

Lemma 4 *Let $C_{p|q}$ be a DSS in the first octant,* i.e. $0 \leq y(u(C_{p|q})) < x(u(C_{p|q}))$, *strictly included in the digital curve $C_{i|j}$. If $r(C_{p|q}, C_{q+1}) \leq -1$ (resp. $r(C_{p|q}, C_{q+1}) \geq \|u(C_{p|q})\|_1 - 1$), there does not exist any upper (resp. lower) digital edge $C_{g|h}$ whose first point belongs to $C_{p|q}$ and that strictly contains the first or last upper (resp. lower) leaning point of $C_{p|q}$, denoted by C_l,* i.e. *such that $p \leq g < l < h$.*

Proof

We will only show by contradiction that $C_{g|h}$ cannot be a upper digital edge that strictly contains the last upper leaning point of $C_{p|q}$. The other cases require a similar proof.

Let us assume that a upper digital edge $C_{g|h}$ exists. On the one hand (i), we posit that C_h belongs to $C_{p|q}$, *i.e.* $p \leq g < l < h \leq q$ and on the other hand (ii), we posit that C_h does not belong to $C_{p|q}$, *i.e.* $p \leq g < l \leq q < h$ (fig. 15).

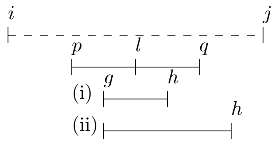


Figure 15: Illustration of the indices used in the proof of lemma 4

The proof is based on the four following arguments, which are proved hereafter :

1. $\rho(u(C_{p|q})) \leq \rho(\overrightarrow{C_g C_l})$
2. $\rho(u(C_{p|q})) > \rho(\overrightarrow{C_l C_h})$
3. $det(\overrightarrow{C_l C_h}, \overrightarrow{C_g C_l}) > 0$
4. $r(C_{g|h}, C_l) > \|u(C_{g|h})\|_1 - 1$ (contradiction)

Due to transitivity, 1) and 2) imply 3) and then, 3) implies 4), which raises a contradiction.

Proof of 1) - Due to property 1, $det(u(C_{p|q}), \overrightarrow{C_g C_l})$ is equal to $r(C_{p|q}, C_l) - r(C_{p|q}, C_g)$ and is thus greater than or equal to 0. In other words, $\rho(u(C_{p|q})) \leq \rho(\overrightarrow{C_g C_l})$.

Proof of 2) - (i) Due to property 1, $det(u(C_{p|q}), \overrightarrow{C_l C_h})$ is equal to $r(C_{p|q}, C_h) - r(C_{p|q}, C_l)$ and is thus strictly smaller than 0. As a result, $\rho(u(C_{p|q})) > \rho(\overrightarrow{C_l C_h})$.

(ii) Since C_l is the last upper leaning point of $C_{p|q}$ and $r(C_{p|q}, C_{q+1}) \leq -1$, lemma 3 applies and $\rho(u(C_{p|q})) > \rho(\overrightarrow{C_l C_h})$.

Proof of 3) - (i) Since $C_{p|q}$ is in the first octant, $x(u(C_{p|q}))$, $x(\overrightarrow{C_g C_l})$ and $x(\overrightarrow{C_l C_h})$ are positive.

(ii) Since $C_{p|q}$ is in the first octant and $C_{i|j}$ is a digital curve, the points of $C_{p|q+1}$ have increasing x-coordinates. Moreover, the points of $C_{g|h}$ have also increasing x-coordinates, because the intersection between $C_{p|q}$ and $C_{g|h}$ contains at least two points, $C_{p|p+1}$. Therefore, $x(u(C_{p|q}))$, $x(u(\overrightarrow{C_g C_l}))$ and $x(\overrightarrow{C_l C_h})$ are positive.

Due to transitivity, from 1) and 2), $\rho(\overrightarrow{C_g C_l}) > \rho(\overrightarrow{C_l C_h})$, which is equivalent to $det(\overrightarrow{C_l C_h}, \overrightarrow{C_g C_l}) > 0$.

Proof of 4) - Due to property 1, $r(C_{g|h}, C_l) - r(C_{g|h}, C_g)$ is equal to $det(u(C_{g|h}), \overrightarrow{C_g C_l})$, which is proportional to $det(\overrightarrow{C_g C_h}, \overrightarrow{C_g C_l})$.

However, $det(\overrightarrow{C_g C_h}, \overrightarrow{C_g C_l})$, which is equal to $det(\overrightarrow{C_l C_h}, \overrightarrow{C_g C_l})$, is strictly greater than 0 due to 3). As a result, $r(C_{g|h}, C_l) > \|u(C_{g|h})\|_1 - 1$, which raises a contradiction. $\blacksquare$

We can now state the main theorem of the subsection:

Theorem 3 *Any convex (resp. concave) digital curve $C_{i|j}$ admits a unique upper (resp. lower) hull.*

Proof

We will prove by induction that any digital convex curve has a unique upper hull. A similar proof can be derived for the lower hull of concave digital curves.

Let the property P_i be "$C_{k|j}$ admits a unique upper hull for all $k \in i, \ldots, j$". The property P_{j-1} is obviously true, because $C_{j-1|j}$ is one small upper digital edge. We will show that P_i is true when P_{i+1} is assumed to be true for some $i \in 1, \ldots, j-2$.

Let p be the greatest index such that C_i is the first upper leaning point of $C_{i|p}$, *i.e.* $C_i = L^{\uparrow}_{min}(C_{i|p})$. Such an index always exists and is unique. Let us assume without loss of generality that $C_{i|p}$ is in the first octant, *i.e.* $0 \leq y(u(C_{i|p})) < x(u(C_{i|p}))$. Let C_l be the last upper leaning point of $C_{i|p}$, *i.e.* $C_l = L^{\uparrow}_{max}(C_{i|p})$ (fig. 16).

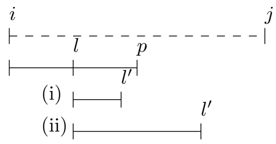


Figure 16: Illustration of the indices used in the proof of theorem 3

Since l is strictly greater than i, $C_{l|j}$ admits a unique upper hull due to the induction hypothesis. On the other hand, $C_{i|l}$ is by definition a digital edge.

existence: In order to prove that $C_{i|j}$ admits a upper hull, it remains to show that the slope of $C_{i|l}$, which is equal by definition to the one of $C_{i|p}$, is strictly smaller than the slope of the first upper digital edge of $C_{l|j}$, denoted by $C_{l|l'}$.

(i) If $l' \leq p$, $C_{l'}$ belongs to $C_{i|p}$. Due to property 1, $det(u(C_{i|p}), \overrightarrow{C_l C_{l'}}) = r(C_{i|p}, C_{l'}) - r(C_{i|p}, C_l)$. Since C_l is the last upper leaning points of $C_{i|p}$, $r(C_{i|p}, C_l) = \|C_{i|p}\|_1 - 1$ and $r(C_{i|p}, C_{l'}) < \|C_{i|p}\|_1 - 1$. Therefore $det(u(C_{i|p}), \overrightarrow{C_l C_{l'}}) < 0$ and the slope of $C_{l|l'}$ is strictly smaller than the one of $C_{i|p}$.

(ii) If $l' > p$, $C_{l'}$ does not belong to $C_{i|p}$. The remainder of C_{p+1} with respect to $C_{i|p}$ cannot belong to the range $0, \dots, \|C_{i|p}\|_1$ because if it is, the first leaning point of $C_{i|p+1}$ would be C_i, which contradicts the maximality of p. If $C_{i|p+1}$ is a DSS, the remainder of C_{p+1} with respect to $C_{i|p}$ is equal to -1. Otherwise, $C_{i|p}$ is maximal at the front and the remainder of C_{p+1} with respect to $C_{i|p}$ is either strictly smaller than -1 or greater than $\|C_{i|p}\|_1$. Since $C_{i|j}$ is convex, the slope of $C_{i|p}$ is strictly greater than the one of the next MS. Therefore, due to theorem 2, the remainder of C_{p+1} with respect to $C_{i|p}$ is strictly smaller than -1. Thus, putting all together, the remainder of C_{p+1} with respect to $C_{i|p}$ is smaller than or equal to -1. According to lemma 3, the slope of $C_{l|l'}$ is strictly smaller than the one of $C_{i|p}$ in this case, which proves that $C_{i|j}$ admits a upper hull.

uniqueness: In order to prove the uniqueness of such a hull, it is enough to notice that C_l cannot be strictly included in a digital edge of the upper hull of $C_{i|j}$ due to lemma 4 and is thus a vertex of such hull. $C_{l|j}$ admits a unique upper hull due to the induction hypothesis and $C_{i|l}$ admits a unique upper hull too because it is by definition a digital edge. The upper hull of $C_{i|j}$ is thus unique. ∎

In the next section we will show that the upper (resp. lower) hull of any convex (resp. concave) digital curve can be retrieved in the course of the cover computation.

4.3 Computation of upper and lower hulls

Theorem 4 gives the smallest digital pattern that is enough to consider to retrieve a vertex or an edge of the upper (resp. lower) hull of any convex (resp. concave) digital curve.

Theorem 4 *Let $C_{k|l}$ be a DSS included in a convex (resp. concave) digital curve $C_{i|j}$. If $r(C_{k|l}, C_{k-1}) \leq -1$ (resp. $r(C_{k|l}, C_{k-1}) \geq \|u(C_{k|l})\|_1$) or $k = i$ and $r(C_{k|l}, C_{l+1}) \leq -1$ (resp. $r(C_{k|l}, C_{l+1}) \geq \|u(C_{k|l})\|_1$) or $l = j$, then the first and last upper (resp. lower) leaning points of $C_{k|l}$ are vertices of the upper (resp. lower) hull of $C_{i|j}$. As a corollary, the part bounded by the first and last upper (resp. lower) leaning points of $C_{k|l}$ (when it is not empty as in fig. 17) is a upper (resp. lower) digital edge.*

Proof

Let us assume that $C_{i|j}$ is convex. A similar reasoning leads to the proof about concave digital curves. Moreover, let us focus on the last upper leaning point of $C_{k|l}$, denoted by C_h, because the same applies for its first upper leaning point. Fig. 17 depicted the configuration of the proof.

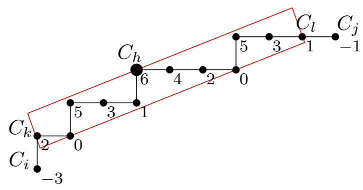


Figure 17: Illustration of theorem 4. The upper leaning point of $C_{k|l}$, C_h is a vertex of the upper hull of the convex part $C_{i|j}$, because $r(C_{k|l}, C_{k-1})$ and $r(C_{k|l}, C_{l+1})$ are less than or equal to -1.

Given that $C_{i|j}$, which is assumed to be convex, admits a unique upper hull due to theorem 3, we will prove by contradiction that there does not exist any upper digital edge $C_{p|q}$ that strictly contains C_h, *i.e.* such that $p < h < q$.

Let us assume that such a upper digital edge exists. Due to the assumption about the remainders of C_{k-1} and C_{l+1} with respect to $C_{k|l}$, $C_{k'-1|l+1}$ is not a DSS. Since $C_{p|q}$ is a DSS, either C_p or C_q belongs to $C_{k|l}$.

Due to lemma 4, there does not exist any upper digital edge $C_{p|q}$ such that C_p belongs to $C_{k|l}$ whereas C_q does not. Similarly, there does not exist any upper digital edge $C_{p|q}$ such that C_q belongs to $C_{k|l}$ whereas C_p does not.

As a consequence, the last upper leaning point of $C_{k|l}$ cannot be strictly contained by any upper digital edge and similarly, the first upper leaning point of $C_{k|l}$ cannot be strictly contained by any upper digital edge. As a consequence, the first and last upper leaning point of $C_{k|l}$ match to one vertex of the upper hull of $C_{i|j}$ if they are confounded or bound a upper digital edge otherwise. ∎

Theorem 4 leads to algorithm 2 that focuses on the upper hull computation of a convex digital curve. To compute the lower hull of a concave digital curve it is enough to replace ↑ with ↓.

In the course of the cover computation, the current DSS $C_{k|l-1}$, is either maximal, maximal at the front, maximal at the back or is the common part of two MSs. In the latter case, $r(C_{k|l-1}, C_{k-1}) = -1$ and $r(C_{k|l-1}, C_l) = -1$ due to corollary 1. The first and last upper leaning points of $C_{k|l-1}$ are vertices of the upper hull due to theorem 4.

When points are added to the front of $C_{k|l-1}$, $C_{k|l-1}$ is maximal at the back, *i.e.* $r(C_{k|l-1}, C_{k-1}) < -1$. If $r(C_{k|l-1}, C_l) = -1$, $C_{k|l}$ is still a DSS but the first upper leaning point get the location of the last one (table 1, column D, row

6). Due to theorem 4, the first and last upper leaning points of $C_{k|l-1}$ are vertices of the upper hull. If they are not confounded, the first upper leaning point of $C_{k|l}$, which is equal to the last upper leaning point of $C_{k|l-1}$, is stored as a new vertex of the upper hull (lines 7 and 20). A similar process is performed when points are removed from the back of the current DSS.

Thus, either during the adding step (lines 5-8, 18-21) or the removing step (lines 12-15, 22-25), any two consecutive retrieved points are the first and last point of a digital edge. Tricky issues occur when the removing step follows the adding step and conversely.

In the first case, the last leaning point of the current DSS, which is maximal, may be not retrieved if the first and last leaning points are distinct, because only the first leaning point has been stored during the adding step and only the new location of the last leaning point will be stored during the removing step that follows. We added lines 10-11 to fix it.

In the second case, the last leaning point of the current DSS, which is the common part of two consecutive MSs, may be retrieved twice, because the last leaning point has been stored during the removing step and will be stored again during the adding step that follows if the first and last leaning points are distinct. We added thus lines 16-17 to fix it.

For all these reasons, algorithm 2 correctly retrieves all the vertices of the upper hull and only them.

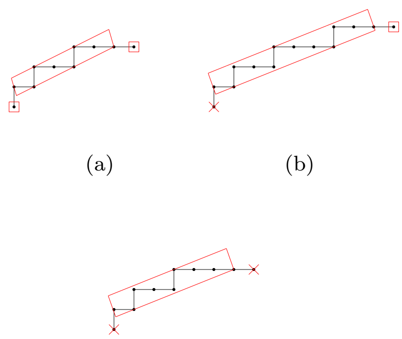
MSs are not enough to compute the upper or lower hull of a respectively convex or concave digital curve. A smaller digital pattern is often required. In convex digital curves, the digital pattern that is enough to consider corresponds to DSSs bounded by two points of remainder smaller than or equal to -1:

- common parts of two consecutive MSs that have two bounding points whose remainder is equal to -1 (lemma 1) (fig. 18.a).
- DSSs maximal at the front or at the back having one bounding point of remainder -1 and the other of remainder strictly less than -1 (theorem 2) (fig. 18.b).
- MSs that have two bounding points whose remainder is strictly less than -1 (theorem 2) (fig. 18.c).

Algorithm 2: Upper hull computation of a convex digital curve

Input: a convex digital curve $C_{i|j}$
Output: the list $\mathcal{L}$ of the vertices of its upper hull

1 $\mathcal{L} \leftarrow \emptyset$; /* list of the vertices of the upper hull */
2 $k \leftarrow i$; /* beginning of the current segment */
3 $l \leftarrow i+1$; /* end of the current segment */
/* initialization */
4 $\mathcal{L} \leftarrow C_k$;
5 **while** $l \leq j$ *and* $C_{k|l}$ *is a DSS* **do**
6   **if** $L^{\uparrow}_{min}(C_{k|l}) \neq L^{\uparrow}_{min}(C_{k|l-1})$ **then**
7     $\mathcal{L} \leftarrow \mathcal{L} + L^{\uparrow}_{min}(C_{k|l})$; /* store this point */
8   $l \leftarrow l+1$; /* add a point to the front */
/* body */
9 **while** $l \leq j$ **do**
  /* store the last leaning point if the first and last leaning points of the MS are not confounded */
10   **if** $L^{\uparrow}_{min}(C_{k|l-1}) \neq L^{\uparrow}_{max}(C_{k|l-1})$ **then**
11     $\mathcal{L} \leftarrow \mathcal{L} + L^{\uparrow}_{max}(C_{k|l-1})$;
12   **while** $C_{k|l}$ *is not a DSS* **do**
13     $k \leftarrow k+1$; /* remove a point from the back */
14   **if** $L^{\uparrow}_{max}(C_{k|l-1}) \neq L^{\uparrow}_{max}(C_{k-1|l-1})$ **then**
15       $\mathcal{L} \leftarrow \mathcal{L} + L^{\uparrow}_{max}(C_{k|l-1})$; /* store */
  /* remove the last leaning point if the first and last leaning points of the common part are not confounded */
16   **if** $L^{\uparrow}_{min}(C_{k|l-1}) \neq L^{\uparrow}_{max}(C_{k|l-1})$ **then**
17     $\mathcal{L} \leftarrow \mathcal{L} - L^{\uparrow}_{max}(C_{k|l-1})$;
18   **while** $l \leq j$ *and* $C_{k|l}$ *is a DSS* **do**
19   **if** $L^{\uparrow}_{min}(C_{k|l}) \neq L^{\uparrow}_{min}(C_{k|l-1})$ **then**
20       $\mathcal{L} \leftarrow \mathcal{L} + L^{\uparrow}_{min}(C_{k|l})$; /* store */
21     $l \leftarrow l+1$; /* add a point to the front */
/* end */
22 **while** $L^{\uparrow}_{max}(C_{k|l-1}) \neq C_j$ **do**
23   $k \leftarrow k+1$; /* remove a point from the back */
24   **if** $L^{\uparrow}_{max}(C_{k|l-1}) \neq L^{\uparrow}_{max}(C_{k-1|l-1})$ **then**
25     $\mathcal{L} \leftarrow \mathcal{L} + L^{\uparrow}_{max}(C_{k|l-1})$; /* store this point */
26 **return** $\mathcal{L}$;



(a) (b)

(c)

Figure 18: Digital patterns containing at least one vertex of the upper hull of a convex digital curve: (a) common parts of two consecutive MSs, (b) DSSs maximal at the front or at the back, (c) MSs. The points marked by a square have a remainder equals to -1, whereas those marked by a cross have a remainder strictly lower than -1.

5 Applications

In this section, we will described several applications of the previous results: convex hull computation (section 5.1), faithful polygonal representation (section 5.2) and minimum perimeter polygon computation (section 5.3).

5.1 Convex hull computation

Let us now consider a closed digital curve C. Let us further assume that C is convex, *i.e.* any open digital curve drawn from C is convex according to definition 5.

Let $C_{1|n+1}$ be the open digital curve associated to C such that C_{n+1} is the repetition of C_1 after one scan of C. Notice that $C_{1|n+1}$ is convex because C is convex (but $C_{1|n+1}$ can also be convex while C is not convex).

The first point C_1 is assumed to be the point of C having the smallest x-coordinate and greatest y-coordinate. The next points $C_2, \dots, C_n$ are assumed to be numbered according to a clockwise orientation.

Let $C_{k_0}, C_{k_1}, \dots, C_{k_M}$ with $1 = k_0 < k_1 < \dots < k_M = (n+1)$ be the sequence of vertices of the upper hull of $C_{1|n+1}$.

Let us now consider the Euclidean convex hull of C. Since C_1 and C_{n+1} are the points having the smallest x-coordinate and greatest y-coordinate, they are vertices of the Euclidean convex hull of C.

The goal of this subsection is to show that retrieving the vertices of the upper hull of $C_{1|n+1}$ provides a way of computing the vertices of the Euclidean convex hull of C.

Theorem 5 *Let C be a convex and closed digital curve. Let $C_{1|n+1}$ be the open digital curve associated to C such that C_1 is a vertex of the Euclidean convex hull of C. The vertices of the upper hull of $C_{1|n+1}$ corresponds to the vertices of the Euclidean convex hull of C.*

Proof

We will prove theorem 5 by induction.

Let the property P_m be "vertex C_{k_m} of the upper hull of $C_{1|n+1}$ is a vertex of the Euclidean convex hull of C" for some $m \in 1, \dots, M$. Property P_1 is true, because C_1 is a vertex of the upper hull of $C_{1|n+1}$ and also a vertex of the Euclidean convex hull of C.

We will prove that P_m is true when P_{m-1} is assumed to be true for some $m \in 2, \dots, M$. In other words, we will prove that C_{k_m} is a vertex of the Euclidean convex hull of C, *i.e.* for all $k \in 1, \dots, n$, $det(\overrightarrow{C_{k_{m-1}}C_{k_m}}, \overrightarrow{C_{k_{m-1}}C_k}) \leq 0$, when $C_{k_{m-1}}$ is assumed to be a vertex of the Euclidean convex hull of C.

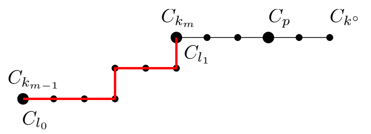
We will first deal with the range $k_{m-1}, \dots, k_m$ (i) and next with the range $1, \dots, n$ minus $k_{m-1}, \dots, k_m$ (ii).

(i) Since $C_{k_{m-1}|k_m}$ is a upper digital edge, for all $k^\bullet \in k_{m-1}, \dots, k_m$, we have $det(\overrightarrow{C_{k_{m-1}}C_{k_m}}, \overrightarrow{C_{k_{m-1}}C_{k^\bullet}}) \leq 0$.

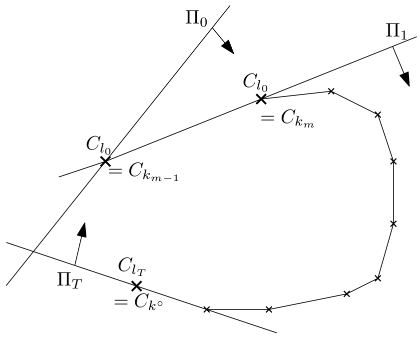
(ii) For all $k^\circ \in 1, \dots, n$ minus $k_{m-1}, \dots, k_m$, let us denote by $C_{k_{m-1}|k^\circ}$ the open digital curve scanned from $C_{k_{m-1}}$ to C_{k° in a clockwise orientation. Since C is assumed to be convex, $C_{k_{m-1}|k^\circ}$ is convex and has a unique upper hull due to theorem 3. Let $C_{l_0}, C_{l_1}, \dots, C_{l_T}$ with $k_{m-1} = l_0 < l_1 < \dots < l_T = k^\circ$ be the sequence of vertices of the upper hull of $C_{k_{m-1}|k^\circ}$.

Let p be the greatest index such that $C_{k_{m-1}}$ is the first upper leaning point of $C_{k_{m-1}|p}$, *i.e.* $C_{k_{m-1}} = L^{\uparrow}_{min}(C_{k_{m-1}|p})$. Since $C_{k_{m-1}|k_m}$ is a upper digital edge, p is greater than k_m and C_{k_m} is the last upper leaning point of $C_{k_{m-1}|p}$, *i.e.* $C_{k_m} = L^{\uparrow}_{max}(C_{k_{m-1}|p})$. As in the proof of theorem 3, the remainder of C_{p+1} with respect to $C_{k_{m-1}|p}$ is smaller than or equal to -1 due to the convexity of C. Therefore, C_{k_m} is the vertex following $C_{k_{m-1}}$ in the upper hull of $C_{k_{m-1}|k^\circ}$ due to theorem 4 and is thus equal to C_{l_1} (fig. 19.a).

For all $t \in 1, \dots, T$, let Π_t be the half-plane defined by $C_{l_{t-1}}$ and C_{l_t} such that $\Pi_t = \{P \in \mathbb{R}^2 | det(\overrightarrow{C_{l_{t-1}}C_{l_t}}, \overrightarrow{C_{l_{t-1}}P}) \leq 0\}$. Since the slopes of



(a)



(b)

Figure 19: Illustration of the proof of theorem 5. (a) $C_{k_{m-1}|k_m}$ (in red) is a digital edge of the upper hull of $C_{1|n+1}$, but also the first digital edge of the upper hull of $C_{k_{m-1}|k^\circ}$ for all $k^\circ \in 1, \dots, n$ minus $k_{m-1}, \dots, k_m$. (b) C_{k° belongs to the intersection of the set of half-planes $\{\Pi_t, t \in [0;T]\}$ and especially to Π_1 whose boundary is the straight line passing through $C_{k_{m-1}}$ and C_{k_m}.

two consecutive upper digital edges of a upper hull are strictly increasing, $C_{l_{t+1}}$ must belong to Π_t.

Moreover, two facts make the location of $C_{l_{t+1}}$ even more constrained. On the one hand, due to the induction hypothesis, $C_{k_{m-1}} = C_{l_0}$ is a vertex of the Euclidean convex hull of C. As a consequence, there exists some vector $\vec{v}$ such that each point of C belongs to the half-plane denoted by Π_0 and defined by $C_{k_{m-1}}$ and $\vec{v}$ as follows: $\Pi_0 = \{P \in \mathbb{R}^2 | det(\vec{v}, \overrightarrow{C_{k_{m-1}}P}) \leq 0\}$. Thus, $C_{l_{t+1}}$ must belong to Π_0 too. On the other hand, the vertices of the upper hull of $C_{k_{m-1}|k^\circ}$ form a simple polygonal line (without any auto-intersection) because $C_{k_{m-1}|k^\circ}$ is a simple digital curve and as a consequence, $C_{l_{t+1}}$ must finally belong to $\cap_{\tau \in [0;t]} \Pi_\tau$ (fig. 19.b). The point $C_{k^\circ} = C_{l_T}$ belongs to $\cap_{t \in [0;T]} \Pi_t$ and especially to Π_1, *i.e.* $det(\overrightarrow{C_{k_{m-1}}C_{l_1}}, \overrightarrow{C_{k_{m-1}}C_{k^\circ}}) \leq 0$. We can thus conclude that $det(\overrightarrow{C_{k_{m-1}}C_{k_m}}, \overrightarrow{C_{k_{m-1}}C_{k^\circ}}) \leq 0$, because $C_{k_m} = C_{l_1}$. ∎

Due to theorem 5, computing the vertices of the Euclidean convex hull of C can be made by retrieving the vertices of the upper hull of $C_{1|n+1}$ provided that C is convex and C_1 is a vertex of Euclidean convex hull of C. Therefore, algorithm 2 can also be viewed as a convex hull algorithm for convex closed digital curves. It has quite interesting features because it is on-line, linear-time and only requires a constant working space if the input is read from an input stream and the output is written to an output stream.

5.2 Faithful polygonal representation

Eckhardt and Dorksen-Reiter [34, 16, 35] looked for a reversible polygon $\mathcal{P}(S)$ that faithfully represent a connected digital set S.

They looked for polygons such that:

- The vertices of $\mathcal{P}(S)$ belong to S.
- S is the Gauss digitization of $\mathcal{P}(S)$, *i.e.* $S = \mathcal{P}(S) \cap \mathbb{Z}^2$.
- $\mathcal{P}(S)$ has as many maximal sequences of consecutive edges of decreasing or increasing slopes as the boundary of S has maximal convex and concave parts.

It turns out that even if some digital sets admit such polygons (think about the Euclidean convex hull of convex digital sets for instance), it is impossible to meet the three requirements at the same time for any sets S [34]. However, we will see in this section that if we replace the second requirement by a weaker one, a faithful polygonal representation always exists. Moreover, we will present an online and linear-time algorithm that computes such representation.

Definition 9 (Faithful representation) *A simple polygonal line $\mathcal{P}$ is a* faithful polygonal representation *of a digital curve $C_{i|j}$ if and only if:*

- *The vertices of $\mathcal{P}$ belong to $C_{i|j}$.*
- *$\mathcal{P}$ is reversible,* i.e. *$C_{i|j}$ can be entirely retrieved from the list of the vertices of $\mathcal{P}$.*
- *Each maximal sequence of consecutive edges of $\mathcal{P}$ with decreasing (resp. increasing slopes) is contained in a maximal convex (resp. concave) part of $C_{i|j}$ and each maximal convex (resp. concave) part of $C_{i|j}$ contains a maximal sequence of consecutive edges of $\mathcal{P}$ with decreasing (resp. increasing) slopes.*

From definitions 7, 8 and theorem 3, any convex (resp. concave) digital curve $C_{i|j}$ admits a convex (resp. concave) polygonal line that is obtained by linking the vertices of the upper (resp. lower) hull of $C_{i|j}$ by straight line segments. The first and third requirements are obviously met. The second one is met too, because the part of $C_{i|j}$ lying between two consecutive vertices is a upper (resp. lower) digital edge, which can be retrieved by the floor (resp. ceil) digitization of the straight line segment joining the two vertices. As a consequence, any strictly convex or concave digital curve has a faithful polygonal representation. Besides, if $C_{1|n+1}$ is the open digital curve associated to a convex closed digital curve C, the polygonal line where all consecutive vertices of the upper hull of $C_{1|n+1}$ are linked by straight line segments is the boundary of the Euclidean convex hull of C due to theorem 5, provided that C_1 is a vertex of that convex hull.

If C is not convex, we can independently compute the upper (resp. lower) hull of each maximal convex (resp. concave) part, because definition 9 is suitable for open digital curves. Tricky issues occur in inflection parts, which belong to both a maximal convex part and a maximal concave part. Though, such parts are MSs and are thus well arithmetically defined. We will see below that the upper and lower hull of two consecutive maximal convex and concave parts can be linked by a straight line that passes through the first upper (resp. lower) and last lower (resp. upper) leaning point of the MS of inflection.

Definition 10 (Faithful polygon) *Let $C_{i^1|j^1}, C_{i^2|j^2}, \dots, C_{i^\Lambda|j^\Lambda}$ be the sequence of the maximal convex or concave parts of a digital curve $C_{i|j}$. The* faithful polygon *(FP) of $C_{i|j}$ is defined by the concatenation of a subsequence of the vertices of the upper or lower hull of the maximal convex or concave parts of $C_{i|j}$:*

$$\odot_{\lambda \in 1,\dots,\Lambda}(C_{k^\lambda_\mu}, C_{k^\lambda_{\mu+1}}, \dots, C_{k^\lambda_\nu})$$

such that:

- *If $C_{i^\lambda|j^\lambda}$ is convex, $C_{k^\lambda_\mu}, C_{k^\lambda_{\mu+1}}, \dots, C_{k^\lambda_\nu}$ are the consecutive vertices of the upper hull of $C_{i^\lambda|j^\lambda}$ located between:*
  - *$C_{k^\lambda_\mu} = C_i$ if $\lambda = 1$ and $C_{k^\lambda_\mu} = L^\uparrow_{max}(S_{i^\lambda\rightarrow})$ otherwise.*
  - *$C_{k^\lambda_\nu} = C_j$ if $\lambda = \Lambda$ and $C_{k^\lambda_\nu} = L^\uparrow_{min}(S_{\leftarrow j^\lambda})$ otherwise.*
- *If $C_{i^\lambda|j^\lambda}$ is concave, $C_{k^\lambda_\mu}, C_{k^\lambda_{\mu+1}}, \dots, C_{k^\lambda_\nu}$ are the consecutive vertices of the lower hull of $C_{i^\lambda|j^\lambda}$ located between:*
  - *$C_{k^\lambda_\mu} = C_i$ if $\lambda = 1$ and $C_{k^\lambda_\mu} = L^\downarrow_{max}(S_{i^\lambda\rightarrow})$ otherwise.*
  - *$C_{k^\lambda_\nu} = C_j$ if $\lambda = \Lambda$ and $C_{k^\lambda_\nu} = L^\downarrow_{min}(S_{\leftarrow j^\lambda})$ otherwise.*

Fig. 20 illustrates the notations of definition 10.

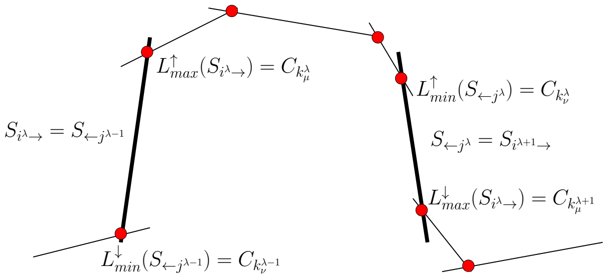


Figure 20: Illustration of definition 10

Theorem 6 *The FP of any digital curve $C_{i|j}$ is a faithful polygonal representation.*

Proof

The first requirement of definition 9 is obviously met. Moreover, the second and third requirements are independently met for all sequences $C_{k^\lambda_\mu}, C_{k^\lambda_{\mu+1}}, \dots, C_{k^\lambda_\nu}$, $\lambda \in 1,\dots,\Lambda$. It remains to show that the FP of $C_{i|j}$ respects its convex and concave parts after the concatenation (i) and that for all $\lambda \in 1,\dots,\Lambda-1$, $C_{k^\lambda_\nu|k^{\lambda+1}_\mu}$ can be retrieved from $C_{k^\lambda_\nu}$ and $C_{k^{\lambda+1}_\mu}$, (ii).

Let us assume that $C_{i^\lambda|j^\lambda}$ is a maximal convex part and $C_{i^{\lambda+1}|j^{\lambda+1}}$ is the maximal concave part that follows for some $\lambda \in 1,\dots,\Lambda-1$. The converse case is similar. Moreover, let us denote by S the MS of inflection $C_{i^{\lambda+1}|j^\lambda}$.

(i) Due to theorem 4, the first and last upper leaning points of $C_{i^{\lambda+1}|j^\lambda}$ are vertices of the upper hull of $C_{i^\lambda|j^\lambda}$ whereas the first and last lower leaning points of $C_{i^{\lambda+1}|j^\lambda}$ are vertices of the lower hull of $C_{i^{\lambda+1}|j^{\lambda+1}}$.

Due to lemma 3, $\rho(\overrightarrow{C_{k^\lambda_{\nu-1}}C_{k^\lambda_\nu}}) > \rho(u(S))$ and by transitivity $\rho(\overrightarrow{C_{k^\lambda_{\nu-1}}C_{k^\lambda_\nu}}) > \rho(\overrightarrow{C_{k^\lambda_\nu}C_{k^{\lambda+1}_\mu}})$. Similarly, $\rho(\overrightarrow{C_{k^\lambda_\nu}C_{k^{\lambda+1}_\mu}}) > \rho(\overrightarrow{C_{k^{\lambda+1}_\mu}C_{k^{\lambda+1}_{\mu+1}}})$.

As a result, $C_{k^\lambda_\mu}, \dots, C_{k^\lambda_\nu}, C_{k^{\lambda+1}_\mu}$ form a polygonal line whose edges are decreasing and that is included in the convex part $C_{i^\lambda|j^\lambda}$ and $C_{k^\lambda_\nu}, C_{k^{\lambda+1}_\mu}, \dots, C_{k^{\lambda+1}_\nu}$ form a polygonal line whose edges are increasing and that is included in the concave part $C_{i^{\lambda+1}|j^{\lambda+1}}$. The third requirement of definition 9 is thus met.

(ii) For all $\lambda \in 1,\dots,\Lambda$ and for all $\delta \in \mu,\dots,\nu-1$, $C_{k^\lambda_\delta|k^\lambda_{\delta+1}}$ is easily retrieved from $C_{k^\lambda_\delta}$ and $C_{k^\lambda_{\delta+1}}$, because $C_{k^\lambda_\delta|k^\lambda_{\delta+1}}$ is a upper digital edge in convex parts and a lower digital edge in concave parts.

However, retrieving $C_{k^\lambda_\nu|k^{\lambda+1}_\mu}$ from $C_{k^\lambda_\nu}$ and $C_{k^{\lambda+1}_\mu}$ for all $\lambda \in 1,\dots,\Lambda-1$ is more tricky because $C_{k^\lambda_\nu|k^{\lambda+1}_\mu}$ is neither a upper digital edge nor a lower digital edge.

For sake of clarity, let us rename $C_{k_\nu^\lambda}$ and $C_{k_\mu^{\lambda+1}}$ into C_p and C_q. Let us consider without loss of generality that $[C_pC_q]$ is in the first quadrant, *i.e.* $x(C_q) - x(C_p)$ and $y(C_q) - y(C_p)$ are positive. Due to definition 10 and by hypothesis, C_p and C_q are respectively the first upper and the last lower leaning points of the same MS, denoted here by S. Let L be the image of C_q after a translation by the vector $\vec{s}\,(-1, 1)$ (see fig. 21).

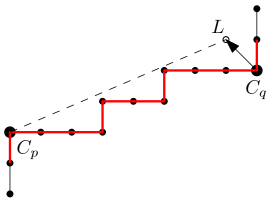


Figure 21: Illustration of the proof of theorem 6: C_p (resp. C_q) is the first upper (resp. last lower) leaning point of the MS of inflection S (in red). The part $C_{p|q-1}$ is the floor digitization of the straight line segment (dashed) joining C_p and L (L excluded), where L is derived from C_q.

The remainder of C_q with respect to $C_{p|q-1}$ is either equal to -1 if C_q is the only lower leaning point of S or equal to 0 otherwise (table 1, column D). Due to property 1, $r(C_{p|q-1}, L) - r(C_{p|q-1}, C_q)$ is equal to $det(u(C_{p|q-1}), \vec{s})$, which is equal to $\|u(C_{p|q-1})\|_1$. The remainder of L with respect to $C_{p|q-1}$ is thus either equal to $\|u(C_{p|q-1})\|_1$ or $\|u(C_{p|q-1})\|_1 - 1$. From table 1, column B, we know that $C_{p|q-1} \cup L$ is a DSS and even more precisely a upper digital edge, because C_p is the first upper leaning point of $C_{p|q-1}$.

Therefore $C_{p|q}$ can be retrieved from C_p and C_q: $C_{p|q-1}$ is indeed the floor digitization of the straight line segment joining C_p and L, but without L. It remains to add C_q to $C_{p|q-1}$ to end the drawing of $C_{p|q}$. ∎

Algorithm 3 computes the FP of a digital curve $C_{i|j}$. Algorithm 1 and 2 have been merged so that the vertices of the upper hull are retrieved in the convex parts while the vertices of the lower hull are retrieved in the concave parts. The outcome of algorithm 3 applied on the wave-shaped digital curve of fig. 8 is shown in fig. 22.

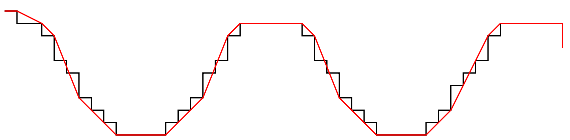


Figure 22: FP of a digital curve in red

Algorithm 3: Faithful polygon of a convex digital curve

Input: a digital curve $C_{i|j}$
Output: The list $\mathcal{L}$ of the vertices of the FP of $C_{i|j}$

$\mathcal{L} \leftarrow \emptyset; \mathcal{L}_u \leftarrow \emptyset; \mathcal{L}_l \leftarrow \emptyset$; /* vertices of the FP */
$k \leftarrow i$; /* beginning of the current segment */
$l \leftarrow i + 1$; /* end of the current segment */
/* initialization */
$\mathcal{L} \leftarrow C_k$;
while $l \leq j$ *and* $C_{k|l}$ *is a DSS* do
    if $L^{\uparrow}_{min}(C_{k|l}) \neq L^{\uparrow}_{min}(C_{k|l-1})$ then
        $\mathcal{L}_u \leftarrow \mathcal{L}_u + L^{\uparrow}_{min}(C_{k|l})$; /* store this point */
    if $L^{\downarrow}_{min}(C_{k|l}) \neq L^{\downarrow}_{min}(C_{k|l-1})$ then
        $\mathcal{L}_l \leftarrow \mathcal{L}_l + L^{\downarrow}_{min}(C_{k|l})$; /* store this point */
    $l \leftarrow l + 1$; /* add a point to the front */
if $l \leq j$ then
    if $r(C_{k|l-1}, C_l) < -1$ then
        $\gamma \leftarrow \uparrow; \mathcal{L} \leftarrow \mathcal{L}_u$; /* convex part */
    else
        $\gamma \leftarrow \downarrow; \mathcal{L} \leftarrow \mathcal{L}_l$; /* concave part */
/* body */
while $l \leq j$ do
    /* store the last leaning point if the first and last leaning points of the MS are not confounded */
    if $L^{\gamma}_{min}(C_{k|l-1}) \neq L^{\gamma}_{max}(C_{k|l-1})$ then
        $\mathcal{L} \leftarrow \mathcal{L} + L^{\gamma}_{max}(C_{k|l-1})$;
    while $C_{k|l}$ *is not a DSS* do
        $k \leftarrow k + 1$; /* remove a point from the back */
        if $L^{\gamma}_{max}(C_{k|l-1}) \neq L^{\gamma}_{max}(C_{k-1|l-1})$ then
            $\mathcal{L} \leftarrow \mathcal{L} + L^{\gamma}_{max}(C_{k|l-1})$; /* store */
    /* remove the last leaning point if the first and last leaning points of the DSS are not confounded */
    if $L^{\gamma}_{min}(C_{k|l-1}) \neq L^{\gamma}_{max}(C_{k|l-1})$ then
        $\mathcal{L} \leftarrow \mathcal{L} - L^{\gamma}_{max}(C_{k|l-1})$;
    while $l \leq j$ *and* $C_{k|l}$ *is a DSS* do
        $l \leftarrow l + 1$; /* add a point to the front */
        if $L^{\gamma}_{min}(C_{k|l}) \neq L^{\gamma}_{min}(C_{k|l-1})$ then
            $\mathcal{L} \leftarrow \mathcal{L} + L^{\gamma}_{min}(C_{k|l})$; /* store this point */
    if $l \leq j$ then
        if $\gamma = \uparrow$ *and* $r(C_{k|l-1}, C_l) \geq \|u(C_{k|l-1})\|_1$ then
            $\gamma \leftarrow \downarrow$; /* from convex to concave */
        else if $\gamma = \downarrow$ *and* $r(C_{k|l-1}, C_l) < -1$ then
            $\gamma \leftarrow \uparrow$; /* from concave to convex */
/* end */
while $L^{\gamma}_{max}(C_{k|l-1}) \neq C_j$ do
    $k \leftarrow k + 1$; /* remove a point from the back */
    if $L^{\gamma}_{max}(C_{k|l-1}) \neq L^{\gamma}_{max}(C_{k-1|l-1})$ then
        $\mathcal{L} \leftarrow \mathcal{L} + L^{\gamma}_{max}(C_{k|l-1})$; /* store this point */
return $\mathcal{L}$;

Note that a digital curve always admits a FP. However, an arbitrary polygon is generally not the FP of a digital curve. Indeed, the FP of a digital curve has angles lying in a small range: $[-\frac{\pi}{2}; \frac{\pi}{2}] \setminus 0$.

Characterizing the FP of a digital curve requires the introduction of the function *quadrant*, which returns true if an edge $\vec{e}$ lies in a given quadrant and false otherwise:

Definition 11 (quadrant) *Let us denote by q the function $q : \mathbb{Z}^2, \{1, 2, 3, 4\} \rightarrow \{true, false\}$ defined by:*

- *$q(\vec{e}, 1) = true$ if $x(\vec{e}) \geq 0$ and $y(\vec{e}) \geq 0$ and false otherwise.*
- *$q(\vec{e}, 2) = true$ if $x(\vec{e}) \geq 0$ and $y(\vec{e}) \leq 0$ and false otherwise.*
- *$q(\vec{e}, 3) = true$ if $x(\vec{e}) \leq 0$ and $y(\vec{e}) \leq 0$ and false otherwise.*
- *$q(\vec{e}, 4) = true$ if $x(\vec{e}) \leq 0$ and $y(\vec{e}) \geq 0$ and false otherwise.*

Proposition 1 *Any two consecutive edges $\vec{e}$ and $\vec{e}'$ of the FP of a digital curve C lie in the same quadrant,* i.e. *$q(\vec{e}, x)$ and $q(\vec{e}', x)$ are true for some $x \in \{1, 2, 3, 4\}$.*

Proof

Let us assume without loss of generality that $\vec{e}$ and $\vec{e}'$ lie in a convex part, *i.e.* $\rho(\vec{e}) > \rho(\vec{e}')$. Moreover, let us assume without loss of generality that $q(\vec{e}, 1)$ is true. By hypothesis, $q(\vec{e}', 2)$ cannot be true and we will show below that C cannot be simply 4-connected if $q(\vec{e}', 3)$ is true or if $q(\vec{e}', 4)$ is true.

- If $q(\vec{e}', 3)$ is true, the first two steps of the digitization of $\vec{e}'$ are $(-1, 0)$ and $(0, -1)$, while the last two steps of the digitization of $\vec{e}$ are $(1, 0)$ and $(0, 1)$.
- If $q(\vec{e}', 4)$ is true, the first step of the digitization of $\vec{e}'$ is $(0, -1)$, while the last step of the digitization of $\vec{e}$ is $(0, 1)$.

In the two above cases, C crosses twice the same point and is thus not simply connected. By contradiction, $q(\vec{e}', 1)$ must be true, which concludes the proof. ∎

We will use definition 11 and proposition 1 in the following subsection in order to define the minimum perimeter polygon of a dilatation of C from the FP of C.

5.3 Minimum Perimeter Polygon

Let us consider a closed digital curve C and let $\mathcal{P}(C)$ be the FP of C. Let the sequence $(p_k \in \mathbb{Z}^2)_{k=1...m}$ be the vertices of $\mathcal{P}(C)$. The indices are assumed to be taken modulo m, the number of vertices.

Let $C^{\square}$ be the dilatation of C by the closed unit square centred on $(0, 0)$, *i.e.* $C^{\square} = C \oplus [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Let us denote by $C_k^{\square}$ the closed unit square centred on C_k and $C_{i|j}^{\square}$ the successive squares from $C_i^{\square}$ to $C_j^{\square}$. Since $C_{i|j}$ is 4-connected, $C_{i|j}^{\square}$ is edge-connected, *i.e.* for all $k \in i, \dots, j-1$, $C_k^{\square}$ and $C_{k+1}^{\square}$ share exactly one edge.

Definition 12 (MPP) *Let C be a closed digital curve and $C^{\square}$ its dilatation by the centred unit square. Let $\mathcal{S}$ be the set of planar curves that go through $C^{\square}$,* i.e. *that completely lie in $C^{\square}$ and intersect each square $C_k^{\square}$ for all $k \in 1, \dots, n$.* A *minimum perimeter polygon (MPP) of $C^{\square}$ is a planar curve $\mathcal{P}^{\star} \in \mathcal{S}$ such that:*

$$\mathcal{P}^{\star} = \arg\min_{\mathcal{P} \in \mathcal{S}} Perimeter(\mathcal{P})$$

The equation of definition 12 has been proven to have a unique solution that is a piecewise linear curve [36]. However, as noted in [37], there is infinitely many polygons having the same perimeter if their vertices are allowed to be collinear. That's why we use the phrase "a MPP" instead of "the MPP", which is the minimum perimeter polygon having no collinear vertices.

The goal of this section is to show that the image of the FP of C after a simple transformation called ϕ-transform, is a MPP of $C^{\square}$. The following definition is based on definition 11 about the quadrant of an edge and proposition 1:

Definition 13 (ϕ-transform) *Let us denote by ϕ the application that maps any FP of C whose vertices are given by the sequence $(p_k)_{k=1...m}$ into a polygon of the half-integer plane $(\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2})$ whose vertices are given by the sequence $(p_k + \vec{v}_k)_{k=1...m}$ with, for all $k \in 1, \dots, m$:*

- *$\vec{v}_k = (\frac{1}{2}, -\frac{1}{2})$ if $\rho(\overrightarrow{p_{k-1}p_k}) > \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 1)$ and $q(\overrightarrow{p_kp_{k+1}}, 1)$ are true or if $\rho(\overrightarrow{p_{k-1}p_k}) < \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 3)$ and $q(\overrightarrow{p_kp_{k+1}}, 3)$ are true.*
- *$\vec{v}_k = (-\frac{1}{2}, -\frac{1}{2})$ if $\rho(\overrightarrow{p_{k-1}p_k}) > \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 2)$ and $q(\overrightarrow{p_kp_{k+1}}, 2)$ are true or if $\rho(\overrightarrow{p_{k-1}p_k}) < \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 4)$ and $q(\overrightarrow{p_kp_{k+1}}, 4)$ are true.*
- *$\vec{v}_k = (-\frac{1}{2}, \frac{1}{2})$ if $\rho(\overrightarrow{p_{k-1}p_k}) > \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 3)$ and $q(\overrightarrow{p_kp_{k+1}}, 3)$ are true or if*

$\rho(\overrightarrow{p_{k-1}p_k}) < \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 1)$ and $q(\overrightarrow{p_kp_{k+1}}, 1)$ are true.

- *$\vec{v}_k = (\frac{1}{2}, \frac{1}{2})$ if $\rho(\overrightarrow{p_{k-1}p_k}) > \rho(\overrightarrow{p_kp_{k+1}})$, $q(\overrightarrow{p_{k-1}p_k}, 4)$ and $q(\overrightarrow{p_kp_{k+1}}, 4)$ are true or if $\rho(\overrightarrow{p_{k-1}p_k}) < \rho(\overrightarrow{p_kp_{k+1}}, 2)$, $q(\overrightarrow{p_{k-1}p_k})$ and $q(\overrightarrow{p_kp_{k+1}}, 2)$ are true.*

In order to show that $\phi(\mathcal{P}(C))$ is a MPP of $C^{\square}$, we give below the points through which a planar curve has to pass in order to be a MPP of $C^{\square}$. Theorem 7 looks like theorem 4:

Theorem 7 *Let $C_{k|l}$ be a DSS strictly included in a convex (resp. concave) digital curve $C_{i|j}$ and assumed to be without loss of generality in the first quadrant. If $r(C_{k|l}, C_{k-1}) \leq -1$ (resp. $r(C_{k|l}, C_{k-1}) \geq \|u(C_{k|l})\|_1$) and $r(C_{k|l}, C_{l+1}) \leq -1$ (resp. $r(C_{k|l}, C_{l+1}) \geq \|u(C_{k|l})\|_1$), then the images of the first and last upper (resp. lower) leaning points of $C_{k|l}$ after a translation by the vector $(\frac{1}{2}, -\frac{1}{2})$ (resp. $(-\frac{1}{2}, \frac{1}{2})$) are vertices of a MPP of $C^{\square}$.*

Proof

We will assume that $C_{i|j}$ is convex, the other case being similar. Moreover, we will focus on the first upper leaning point of $C_{k|l}$ because similar results may be derived for the last upper leaning point.

Let B, E and L be respectively the images of C_{k-1}, C_{l+1} and $L^{\uparrow}_{min}(C_{k|l})$ after a translation by the vector $(\frac{1}{2}, -\frac{1}{2})$. We will use below parametrized translations defined as follows: given a real number of the range $[0, 1]$ denoted by λ, P_λ is the image of P after a translation by the vector $(-\lambda, \lambda)$. We have of course $P = P_0$.

Let $\mathcal{D}$ be the straight line of slope $u(C_{k|l})$ and passing through L. For all $\lambda \in]0, 1]$, L_λ is obviously located strictly above $\mathcal{D}$.

On the other hand, due to property 1, $det(u(C_{k|l}), \overrightarrow{LB})$ and $det(u(C_{k|l}), \overrightarrow{LE})$ are less than or equal to $-\|u(C_{k|l})\|_1$, because $r(C_{k|l}, C_{k-1})$ and $r(C_{k|l}, C_{l+1})$ are less than or equal to -1 by hypothesis. Applying property 1 again, we get that $det(u(C_{k|l}), \overrightarrow{LB_\lambda})$ and $det(u(C_{k|l}), \overrightarrow{LE_\lambda})$ are less than or equal to $(\lambda - 1).\|u(C_{k|l})\|_1$ and thus less than or equal to 0 for all $\lambda \in [0, 1]$. In other words, B_λ and E_λ are located on or below $\mathcal{D}$ for all $\lambda \in [0, 1]$.

Let us now consider the planar curves that start from B_λ for some $\lambda \in [0, 1]$, end at E_λ for some $\lambda \in [0, 1]$, go through $C^{\square}_{k-1|l+1}$ by passing through L_λ for some $\lambda \in]0, 1]$ (but not through $L = L_0$). One of them is depicted fig. 23.

Such curves necessarily cross $\mathcal{D}$ at least once before and after L because B_λ and E_λ are located on or below $\mathcal{D}$ for all $\lambda \in [0, 1]$ and L_λ is strictly located above $\mathcal{D}$ for all $\lambda \in]0, 1]$. Let us denote by I_1 and I_2 the nearest intersection points before and after L. Cutting the curves at I_1 and I_2 and reconnecting them by the straight segment $[I_1 I_2]$ leads to valid curves of smaller length. Therefore, the shortest path that starts from B_λ for some $\lambda \in [0, 1]$, ends at E_λ for some $\lambda \in [0, 1]$ and goes through $C^{\square}_{k-1|l+1}$ cannot pass through L_λ for all $\lambda \in]0, 1]$ and must pass through $L = L_0$, which means that L is a vertex of a MPP of $C^{\square}$. ∎

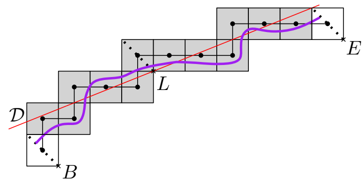


Figure 23: Illustration of the proof of theorem 7 : the dilatation of the DSS $C_{k|l}$ is depicted in gray, and the positions B_λ, E_λ and L_λ for all $\lambda \in [0, 1]$ are depicted with dotted segments. Note that B_λ and E_λ are below the line $\mathcal{D}$ while L_λ is above. Any planar curves (like the one depicted in purple) that go through $C^{\square}_{k-1|l+1}$ by passing through L_λ for some $\lambda \in]0, 1]$ must cross $\mathcal{D}$ at least twice and can thus be shorten. The shortest curve that go through $C^{\square}_{k-1|l+1}$ must therefore pass through $L = L_0$.

Theorem 7 shows that in the first quadrant the first and last upper (resp. lower) leaning points of $C_{k|l}$ translated by $(\frac{1}{2}, -\frac{1}{2})$ (resp. $(-\frac{1}{2}, \frac{1}{2})$) are vertices of a MPP of $C^{\square}$. In the second, third and fourth quadrant, it can be similarly shown that these points are vertices of a MPP of $C^{\square}$ when translated respectively by $(-\frac{1}{2}, -\frac{1}{2})$ (resp. $(\frac{1}{2}, \frac{1}{2})$), $(-\frac{1}{2}, \frac{1}{2})$ (resp. $(\frac{1}{2}, -\frac{1}{2})$) and $(-\frac{1}{2}, -\frac{1}{2})$ (resp. $(\frac{1}{2}, \frac{1}{2})$). Therefore, due to theorems 4 and 7, definitions 10 and 13, each vertex of $\phi(\mathcal{P}(C))$ is also a vertex of a MPP of $C^{\square}$. Moreover, the straight segment joining any two consecutive vertices of $\phi(\mathcal{P}(C))$ is the shortest path going through $C^{\square}$ and joining the two vertices, which proves, with theorem 7, the following theorem:

Theorem 8 *Let C be a closed digital curve and $C^{\square}$ its dilatation by the centred unit square. The ϕ-transform of the FP of C is a MPP of $C^{\square}$.*

A MPP of $C^{\square}$ is thus given by the ϕ-transform of the FP of C, which is computed by algorithm 3 in linear-time with integer-only computations. The ϕ-transform of the FP of fig. 22 is shown in fig. 24.

Several other algorithms have been proposed, but only a few ones have been designed to take profit

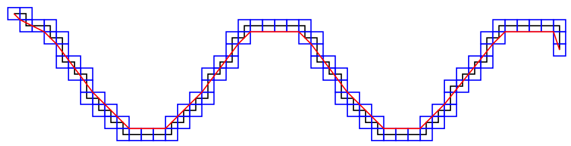
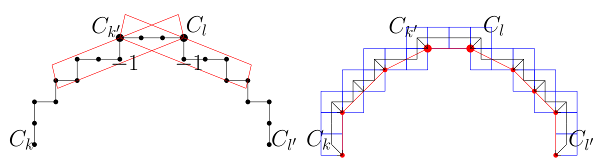


Figure 24: MPP (in red) of the dilatation of a digital curve by the centered unit square (in blue)



(a) (b)

Figure 25: In (a), the last MS of the first quadrant word $C_{k|l}$ and the first MS of the second quadrant word $C_{k'|l'}$ are depicted by a red bounding box. Notice that the first and last point of their common part $C_{k'|l}$ are also its fist and last upper leaning points and that the remainder of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ is equal to -1. In (b), the ϕ-transform of $C_{k'}$ and C_l are two consecutive vertices of a MPP of $C^{\square}$ due to theorems 4 and 7.

of the specificity of the digital curves. One of them [38] does not seem to compute a MPP in all cases as noticed in [15, 39]. Recently, Provençal and Lachaud [15] have independently proposed an arithmetic and a combinatorial algorithm.

The arithmetic one first decomposes the digital curve into convex and concave parts by the means of the slope of the MSs (definition 5) and next decomposes each convex or concave part in *quadrant words*, *i.e.* open digital curves having only two kinds of steps. For each piece of digital curve, their lower or upper convex hull is computed using [40]. A MPP is derived from the union of these convex hulls. The combinatorial algorithm is an on-line way of computing the same polygon. However, the authors does not explain why the partitioning of the digital curve into quadrant words makes sens.

Any partitioning is actually valid if and only if each part begins and ends at a vertex of a MPP. Our previous results prove indeed that each maximal quadrant words begins and ends at a vertex of the FP, whose ϕ-transform is a vertex of a MPP.

Let us consider the intersection $C_{k'|l}$ between two consecutive quadrant words $C_{k|l}$ and $C_{k'|l'}$ in a convex part of C. The part $C_{k'|l}$ is also the intersection between the last MS of the first quadrant word $S_{\leftarrow l}$ and the first MS of the second quadrant word $S_{k'\rightarrow}$ (fig. 25.a) and as a consequence: (i) the first and last upper leaning points of $C_{k|l}$ are its first and last points, $C_{k'}$ and C_l, (ii) due to corollary 1, the remainder of $C_{k'-1}$ and C_{l+1} with respect to $C_{k'|l}$ is equal to -1. Due to theorems 4 and 7, the ϕ-transform of $C_{k'}$ and C_l are thus two consecutive vertices of a MPP of $C^{\square}$ (fig. 25.b), which proves the correctiveness of the algorithms proposed in [15].

The perimeter of a MPP of $C^{\square}$ is known to be a good estimator of the perimeter of C [38, 41, 26, 36] and has been recently used as the internal energy of a digital deformable model [39].

6 Conclusion

The local convexity properties of digital curves have been studied in this paper.

We have arithmetically characterized the smallest digital pattern required for checking convexity in section 3 (theorem 2), which answers to a question raised by Eckhardt [16]. The smallest digital pattern required for checking convexity is actually given by a maximal segment (MS), *plus* at least one of the two points located just before and after this segment.

Moreover, we went further and have arithmetically characterized the smallest digital pattern that contains a vertex of the upper (resp. lower) hull of a given convex (resp. concave) digital curve in section 4 (theorem 4). In convex digital curves, such a digital pattern is actually a digital straight segment bounded by two points whose remainder is less than or equal to -1.

These theoretical results lead to online and linear-time algorithms (algorithms 1, 2 and 3) that only use well-known routines: adding a point to the front of a DSS [30] and removing a point from the back of a DSS [18]. The proposed algorithms have been implemented in C++ and the code is available on the web site of the LIRIS lab[1]. Fig. 8, 22 and 24 are the outputs of the program. They are useful for polygonal representations as shown in section 5: computation of the convex hull of convex digital curves, computation of a faithful polygon (FP) whose existence was another question raised by Eckhardt [16] and computation of the well-known minimum-perimeter polygon (MPP).

The FP and the MPP appear to be complementary to each other. The FP is reversible and exactly reflects the convex and concave parts of the digital curve, whereas the MPP minimizes the number of inflection points required to represent the digital curve and is known to provide good estimators of

[1] http://liris.cnrs.fr/m2disco/index_en.html?softwares

tangent and length.

All the results derived in this paper are based on arithmetics and properties of determinant calculus (see for instance property 1), but recent works show that similar results can be obtained with tools coming from combinatorics on words [42, 15]. It turns out that Christoffel words of the chain code of a digital curve corresponds to digital edges. Also, the factorization of the chain code of a convex digital curve into Lyndon words corresponds to its upper hull. This suggests to study relationships between concepts coming from digital geometry and combinatorics on words, which will benefit to both areas.

Eventually, note that the parts of the digital curve highlighted by our algorithms does not reflect the visual parts of the original shape if the resolution is to high with respect to the scale of its main features or if some stochastic noise is introduced in the digitization. Several ways of copying with this problem can be followed while keeping an arithmetic approach that leads to fast algorithms with integer-only computations: (i) find a deformation process of the digital curve so that it sticks to the expected shape, like in digital deformable models [39], (ii) find a discrete simplification process of the digital curve in the manner of [1] or (iii) work on sub-sampled versions of the initial digital curve as done in [43]. This work and its perspectives lead to think that digital convexity will help to design an efficient and accurate method dedicated to the extraction of visual parts.

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