Quasi-Affine Transformation in Higher Dimension

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1 Introduction

In many computer vision and image processing applications, we are facing new constraints due to the image sizes both in dimension with 3-D and 3-D+t medical acquisition devices, and in resolution with VHR (Very High Resolution) satellite images. This article deals with high performance image transformations using quasi-affine transforms (QATs for short), which can be viewed as a discrete version of general affine transformations. QAT can approximate rotations and scalings, and in some specific cases, QAT may also be one-to-one and onto mappings from $\mathbb{Z}^n$ to $\mathbb{Z}^n$, leading to exact computations.

In dimension 2, the QAT appeared in several articles [1,2,3,4,5]. To summarize the main results, the authors have proved several arithmetical results on QAT in 2-D leading to efficient QAT algorithms. More precisely, thanks to periodic properties of pavings induced by the reciprocal map, the image transformation can be obtained using a set of precomputed canonical pavings. In this paper, we focus on a theoretical analysis of n-dimensional QAT. The idea is to investigate fundamental results in order to be able to design efficient transformation algorithms in dimension 2 or 3. More precisely, we demonstrate the arithmetical and periodic structures embedded in $n-$dimensional QAT.

In Section 2, we first detail preliminary notations and properties. Then, Section 3 contains the main theoretical results leading to a generic n-D transformation algorithm sketched in Section 4. In Sections 5 and 6, we details the QAT algorithms in dimension 2 and 3.

2 Preliminaries

2.1 Notations

Before we introduce arithmetical properties of QAT in higher dimension, we first detail the notations considered in this paper. Let n denotes the dimension of the considered space, V_i denotes the i^{th} coordinate of vector V, and $M_{i,j}$ denotes the $(i, j)^{th}$ coefficient of matrix M. We use the notation $\gcd(a, b, \ldots)$ for the greatest common divisor of an arbitrary number of arguments, and $\mathrm{lcm}(a, b, \ldots)$ for their least common multiple.

Let $\left[\frac{a}{b}\right]$ denotes the quotient of the euclidean division of a by b, that is the integer $q \in \mathbb{Z}$ such that $a = bq + r$ satisfying $0 \leq r < |b|$ regardless of the sign of b[1]. We consider the following generalization to $n-$dimensional vectors:

$$\left[\frac{V}{b}\right] = \begin{pmatrix} \left[\frac{V_0}{b}\right] \\ \vdots \\ \left[\frac{V_{n-1}}{b}\right] \end{pmatrix} \text{ and } \left\{\frac{V}{b}\right\} = \begin{pmatrix} \left\{\frac{V_0}{b}\right\} \\ \vdots \\ \left\{\frac{V_{n-1}}{b}\right\} \end{pmatrix} . \tag{1}$$

2.2 Quasi-Affine Transformation Definitions

Defined in dimension 2 in [1,2,3,4,5], we consider a straightforward generalization to $\mathbb{Z}^n$ spaces.

Definition 1 (QAT). *A quasi-affine transformation is a triple* $(\omega, M, \boldsymbol{V}) \in \mathbb{Z} \times M_n(\mathbb{Z}) \times \mathbb{Z}^n$ *(we assume that* $\det(M) \neq 0$*). The associated application is:*

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ X &\longmapsto \left[\frac{MX + V}{\omega}\right] . \end{aligned}$$

And the associated affine application is:

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ X &\longmapsto \frac{MX + V}{\omega} . \end{aligned}$$

In other words, a QAT is the composition of the associated affine application and the integer part floor function.

Definition 2. *A QAT is said to be contracting if* $\omega^n > |\det(M)|$*, otherwise it is said to be dilating.*

In other words, a QAT is contracting if and only if the associated affine application is contracting. Note that if $\omega^2 = |\det(M)|$, the QAT is dilating, even if the associated affine application is an isometry.

Definition 3. *The inverse of a QAT* (ω, M, V) *is the QAT:*

$$(\det(M), \omega \operatorname{com}(M)^t, -\operatorname{com}(M)^t V) , \tag{2}$$

where ^{t} *denotes the transposed matrix and* $\operatorname{com}(M)$ *the co-factor matrix of* M[2].

The associated affine application of the inverse of a QAT is therefore the inverse of the affine application associated to the QAT. However, due to the nested floor function, the composition $f \cdot f^{-1}$ is not the identity function in the general case.

In Section 6, we have to consider a generalized form of the Bezout identity in dimension 3:

[1] $\left\{\frac{a}{b}\right\}$ denotes the corresponding remainder $\left\{\frac{a}{b}\right\} = a - b\left[\frac{a}{b}\right]$.
[2] Remind that $M \operatorname{com}(M)^t = \operatorname{com}(M)^t M = \det(M) I_n$.

Proposition 1. $\forall(a,b,c) \in \mathbb{Z}^3, \exists(u,v,w) \in \mathbb{Z}^3 / au + bv + cw = \gcd(a,b,c)\,.$

Proof. The proof is given in Sect. A.8.

3 QAT Properties in Higher Dimensions

Without loss of generality, we suppose that the QAT is contracting.

3.1 Pavings of a QAT

A key feature of a QAT in dimension 2 is the paving induced by the reciprocal map of a discrete point. In the following, we adapt the definitions in higher dimensions and prove that a QAT in $\mathbb{Z}^n$ also carries a periodic paving.

Definition 4 (Paving). *Let f be a QAT. For $Y \in \mathbb{Z}^n$, we denote:*

$$P_Y = \{X \in \mathbb{Z}^n / f(X) = Y\} = f^{-1}(Y)\,, \tag{3}$$

P_Y is called order 1 paving of index Y of f.

P_Y can be interpreted as a subset of $\mathbb{Z}^n$ (maybe empty) that corresponds to the reciprocal map of Y by f. We easily show that the set of pavings of a QAT forms a paving of the considered space (see Fig. 1). In dimension 2, this definition exactly coincides with previous ones [1,3,4,5,2].

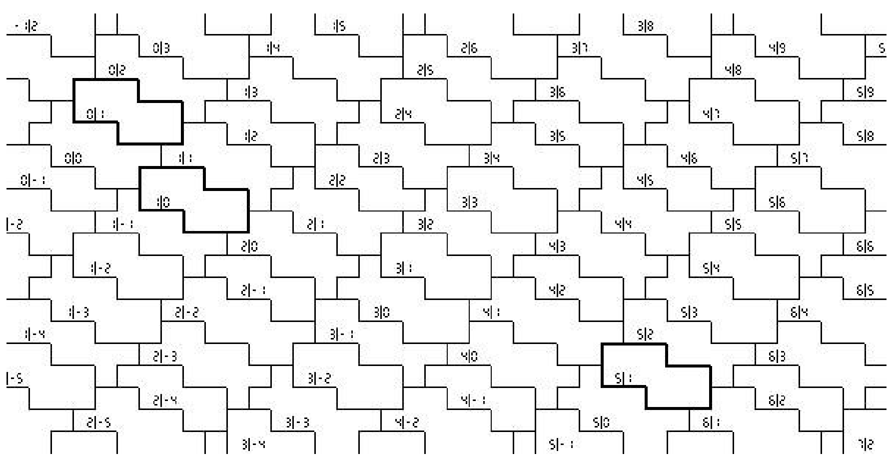


Fig. 1. Pavings of the QAT $\left(84, \begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}, \begin{pmatrix} 150 \\ -500 \end{pmatrix}\right)$ with their indexes (in 2D, these are the couples: $Y = (x,y)$).

Definition 5. *P_Y is said arithmetically equivalent to P_Z (denoted $P_Y \equiv P_Z$) if:*

$$\forall X \in P_Y, \exists X' \in P_Z / \left\{ \frac{MX + V}{\omega} \right\} = \left\{ \frac{MX' + V}{\omega} \right\} . \tag{4}$$

Again, this definition is equivalent (as shown below) to those given in the literature.

Theorem 1. *The equivalence relationship is symmetric,* i.e.*:*

$$P_Y \equiv P_Z \Leftrightarrow P_Z \equiv P_Y . \tag{5}$$

Proof. The proof is given in Sect. A.1.

Figure 1 illustrates arithmetically equivalent pavings: the pavings of index $(0,1)$ and $(5,1)$ are arithmetically equivalent (see Table 1).

$P_{0,1}$		$P_{5,1}$		$P_{1,0}$	
X	$\left\{\frac{MX+V}{\omega}\right\}$	X	$\left\{\frac{MX+V}{\omega}\right\}$	X	$\left\{\frac{MX+V}{\omega}\right\}$
$\begin{pmatrix}3\\15\end{pmatrix}$	$\begin{pmatrix}21\\10\end{pmatrix}$	$\begin{pmatrix}27\\3\end{pmatrix}$	$\begin{pmatrix}21\\10\end{pmatrix}$	$\begin{pmatrix}6\\11\end{pmatrix}$	$\begin{pmatrix}17\\4\end{pmatrix}$
$\begin{pmatrix}5\\14\end{pmatrix}$	$\begin{pmatrix}56\\10\end{pmatrix}$	$\begin{pmatrix}29\\2\end{pmatrix}$	$\begin{pmatrix}56\\10\end{pmatrix}$	$\begin{pmatrix}8\\10\end{pmatrix}$	$\begin{pmatrix}52\\4\end{pmatrix}$
$\begin{pmatrix}4\\15\end{pmatrix}$	$\begin{pmatrix}33\\28\end{pmatrix}$	$\begin{pmatrix}28\\3\end{pmatrix}$	$\begin{pmatrix}33\\28\end{pmatrix}$	$\begin{pmatrix}7\\11\end{pmatrix}$	$\begin{pmatrix}29\\22\end{pmatrix}$
$\begin{pmatrix}6\\14\end{pmatrix}$	$\begin{pmatrix}68\\28\end{pmatrix}$	$\begin{pmatrix}30\\2\end{pmatrix}$	$\begin{pmatrix}68\\28\end{pmatrix}$	$\begin{pmatrix}9\\10\end{pmatrix}$	$\begin{pmatrix}64\\22\end{pmatrix}$
$\begin{pmatrix}3\\16\end{pmatrix}$	$\begin{pmatrix}10\\46\end{pmatrix}$	$\begin{pmatrix}27\\4\end{pmatrix}$	$\begin{pmatrix}10\\46\end{pmatrix}$	$\begin{pmatrix}6\\12\end{pmatrix}$	$\begin{pmatrix}6\\40\end{pmatrix}$
$\begin{pmatrix}5\\15\end{pmatrix}$	$\begin{pmatrix}45\\46\end{pmatrix}$	$\begin{pmatrix}29\\3\end{pmatrix}$	$\begin{pmatrix}45\\46\end{pmatrix}$	$\begin{pmatrix}8\\11\end{pmatrix}$	$\begin{pmatrix}41\\40\end{pmatrix}$
$\begin{pmatrix}7\\14\end{pmatrix}$	$\begin{pmatrix}80\\46\end{pmatrix}$	$\begin{pmatrix}31\\2\end{pmatrix}$	$\begin{pmatrix}80\\46\end{pmatrix}$	$\begin{pmatrix}10\\10\end{pmatrix}$	$\begin{pmatrix}76\\40\end{pmatrix}$
$\begin{pmatrix}4\\16\end{pmatrix}$	$\begin{pmatrix}22\\64\end{pmatrix}$	$\begin{pmatrix}28\\4\end{pmatrix}$	$\begin{pmatrix}22\\64\end{pmatrix}$	$\begin{pmatrix}7\\12\end{pmatrix}$	$\begin{pmatrix}18\\58\end{pmatrix}$
$\begin{pmatrix}6\\15\end{pmatrix}$	$\begin{pmatrix}57\\64\end{pmatrix}$	$\begin{pmatrix}30\\3\end{pmatrix}$	$\begin{pmatrix}57\\64\end{pmatrix}$	$\begin{pmatrix}9\\11\end{pmatrix}$	$\begin{pmatrix}53\\58\end{pmatrix}$
$\begin{pmatrix}5\\16\end{pmatrix}$	$\begin{pmatrix}34\\82\end{pmatrix}$	$\begin{pmatrix}29\\4\end{pmatrix}$	$\begin{pmatrix}34\\82\end{pmatrix}$	$\begin{pmatrix}8\\12\end{pmatrix}$	$\begin{pmatrix}30\\76\end{pmatrix}$
$\begin{pmatrix}7\\15\end{pmatrix}$	$\begin{pmatrix}69\\82\end{pmatrix}$	$\begin{pmatrix}31\\3\end{pmatrix}$	$\begin{pmatrix}69\\82\end{pmatrix}$	$\begin{pmatrix}10\\11\end{pmatrix}$	$\begin{pmatrix}65\\76\end{pmatrix}$

Table 1. Pavings of index $(0,1)$, $(5,1)$ and $(1,0)$ of the QAT $\left(84, \begin{pmatrix}12 & -11\\18 & 36\end{pmatrix}, \begin{pmatrix}150\\-500\end{pmatrix}\right)$

Definition 6. *P_Y and P_Z are said geometrically equivalent if:*

$$\exists \boldsymbol{v} \in \mathbb{Z}^n / P_Y = T_{\boldsymbol{v}} P_Z \,, \tag{6}$$

where $T_{\boldsymbol{v}}$ denotes the translation of vector $\boldsymbol{v}$.

In Figure 1, the pavings of indexes $(0,1)$ and $(1,0)$ are geometrically equivalent. In image processing purposes, when we want to transform a $n-$dimensional image by a QAT, geometrically equivalent pavings will allow us to design fast transformation algorithms.

Theorem 2. *If $P_Y \equiv P_Z$, then P_Y and P_Z are geometrically equivalent. Since $P_Y \equiv P_Z$, there exists $X \in P_Y$ and $X' \in P_Z$ such that:*

$$\left\{ \frac{MX+V}{\omega} \right\} = \left\{ \frac{MX'+V}{\omega} \right\} .$$

Then $\boldsymbol{v} = X - X'$ is the translation vector:

$$P_Y = T_{\boldsymbol{v}} P_Z \,.$$

In dimension 2, this theorem is also proved in [4].

Proof. The proof is given in Sect. A.2

In a computational point of view, if a paving P_Y has been already computed, and if we know that $P_Y \equiv P_Z$, then P_Z can be obtained by translation of P_Y. In Figure 1, the pavings of index $(0,1)$ and $(5,1)$ are arithmetically equivalent (see Table 1), therefore they are geometrically equivalent (as we can check on the figure). Note that the inverse implication is false: in Figure 1, the pavings of index $(0,1)$ and $(1,0)$ are geometrically equivalent but they are not arithmetically equivalent (see Table 1).

3.2 Paving Periodicity

Definition 7. $\forall 0 \leq i < n$, *We define the set $\mathcal{A}_i$ as follows:*

$$\mathcal{A}_i = \{\alpha \in \mathbb{N}^* / \exists (\beta_j)_{0 \leq j < i} \in \mathbb{Z}^i, \forall (y_0, \ldots, y_{n-1}) \in \mathbb{Z}^n, \\ P_{y_0,\ldots,y_i+\alpha,\ldots,y_{n-1}} \equiv P_{y_0+\beta_0,\ldots,y_{i-1}+\beta_{i-1},y_i,\ldots,y_{n-1}}\}$$

Theorem 3 (Perdiodicity). *The set of QAT pavings is $n-$periodic, in other words*

$$\forall 0 \leq i < n, \mathcal{A}_i \neq \emptyset$$

Proof. The proof is given in Sect. A.3.

If we consider $\alpha = |det(M)|$ as in Sect. A.3, we have demonstrated the periodic structure of QAT pavings since $P_Y \equiv P_{Y+\alpha e_i}$ for each i. We investigate now the quantities α_i which are minimal for each dimension i,

Definition 8. *$\forall 0 \leq i < n$, let us consider $\alpha_i = \min(\mathcal{A}_i)$. We define $\{\beta_j^i\}_{0\leq j<i} \in \mathbb{Z}^i$ and $U_i \in \mathbb{Z}^n$ such that*

$$\forall (y_0,\dots,y_{n-1}) \in \mathbb{Z}^n, P_{y_0,\dots,y_i+\alpha_i,\dots,y_{n-1}} = T_{\boldsymbol{U_i}} P_{y_0+\beta_0^i,\dots,y_{i-1}+\beta_{i-1}^i,y_i,\dots,y_{n-1}} .$$

Thanks to Theorem 2 and using notations of Def. 8, let $X \in P_{y_0,\dots,y_i+\alpha_i,\dots,y_{n-1}}$ and $X' \in P_{y_0+\beta_0^i,\dots,y_{i-1}+\beta_{i-1}^i,y_i,\dots,y_{n-1}}$, such that $\left\{\frac{MX+V}{\omega}\right\} = \left\{\frac{MX'+V}{\omega}\right\}$. Then, we have $U_i = X - X'$.

Let us suppose that quantities α_i, β_j^i and U_i are given (see [6] for computation details in dimension 2 and 3). To paraphrase above results, α_i and its associated U_i and $\{\beta_j^i\}$ allows us to *reduce* the i^{th} component of Y while preserving the geometrically equivalence relationship. If we repeat this *reduction* process to each component from $n-1$ down-to 0, we construct a point Y^0 such that P_Y and P_{Y^0} are geometrically equivalent. The following theorem formalizes this principle and define the initial period paving P_{Y^0}.

Theorem 4. *$\forall (y_0,\dots,y_{n-1}) \in \mathbb{Z}^n$, we have $P_{y_0,\dots,y_{n-1}} = T_W P_{y_0^0,\dots,y_{n-1}^0}$ with*

$$W = \sum_{i=0}^{n-1} w_i U_i$$

$$\text{and } \forall n > i \geq 0, \begin{cases} w_i = \left\lceil \frac{y_i + \sum_{j=i+1}^{n-1} w_j \beta_i^j}{\alpha_i} \right\rceil \\ y_i^0 = \left\{ \frac{y_i + \sum_{j=i+1}^{n-1} w_j \beta_i^j}{\alpha_i} \right\} \end{cases} .$$

Proof. The proof is given in Sect. A.4.

Therefore, if we already computed the pavings $P_{y_0^0,\dots,y_{n-1}^0}$ for $0 \leq y_i^0 < \alpha_i$, we can obtain any paving by translation of one of these pavings.

3.3 Super-paving of a QAT

We now describe how to compute these initial period pavings based on the notion of super-paving (see Fig. 2).

Definition 9. *A super-paving of a QAT is the set $\mathcal{P}$ such that*

$$\mathcal{P} = \bigcup_{0\leq Y^0 < (\alpha_0,\dots,\alpha_{n-1})} P_{Y^0}$$

In other words, the super-paving is the union of all pavings of the initial period. In dimension 2, this definition coincides with definitions given in [4,3,5].

Theorem 5. *$\mathcal{P}$ is the paving $P_{(0,\dots,0)}$ of the QAT defined by:*

$$\left(\omega \operatorname{lcm}_{0\leq i<n}(\alpha_i), \begin{pmatrix} \theta_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_{n-1} \end{pmatrix} M, \begin{pmatrix} \theta_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_{n-1} \end{pmatrix} V \right) , \tag{7}$$

with $\forall 0 \leq i < n-1$,

$$\theta_i = \frac{\mathrm{lcm}_{0 \leq j < n-1}(\alpha_j)}{\alpha_i}\,.$$

Proof. The proof is given in Sect. A.5.

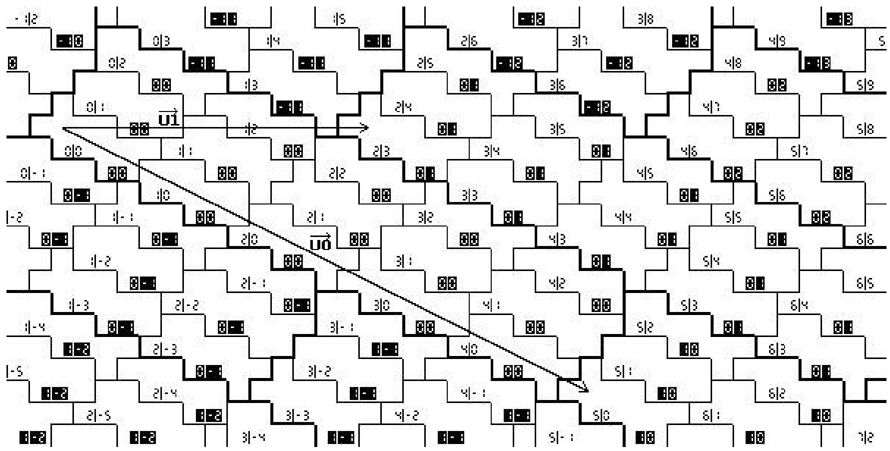


Fig. 2. Super-paving decomposition of the QAT defined in Fig. 1. Arrows illustrate a basis of the periodic structure.

Hence, we can associate a canonical paving to each point of the super-paving. More precisely, the super-paving allows us to compute the equivalence classes for the arithmetical equivalence relationship between two pavings.

3.4 Paving Construction

In this section, we focus on an arithmetic paving construction algorithm. Hence, using the results of the previous section, such a construction algorithm will be used to compute canonical pavings in the super-paving.

Definition 10. *The matrix T is the Hermite Normal Form of the QAT matrix M if:*

- *T is upper triangular, with coefficients $\{T_{ij}\}$ such that $T_{ii} > 0$;*
- *$\exists H \in GL_n(\mathbb{Z}) / MH = T$.*

If M is nonsingular integer matrix, the Hermite Normal Form exists. Note also that if $H \in M_n(\mathbb{Z})$, then $H \in GL_n(\mathbb{Z}) \Leftrightarrow |\det(H)| = 1$.

For example, given $\begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}$, we have:

$$\begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 35 & 12 \\ 0 & 18 \end{pmatrix}\,.$$

Using the Hermite Normal Form, we can design a fast paving computation algorithm formalized in the following theorem:

Theorem 6. *$\forall Y \in \mathbb{Z}^n$, let $MH = T$ be the Hermite Normal Form of the QAT matrix M, then*

$$P_Y = \{HX \ / \ \forall n > i \geq 0, A_i(X_{i+1}, \ldots, X_{n-1}) \leq X_i < B_i(X_{i+1}, \ldots, X_{n-1})\} \tag{8}$$

With

$$A_i(X_{i+1}, \ldots, X_{n-1}) = -\left\lceil \frac{-\omega Y_i + \sum_{j=i+1}^{n-1} T_{i,j} X_j + V_i}{T_{i,i}} \right\rceil ,$$

$$B_i(X_{i+1}, \ldots, X_{n-1}) = -\left\lceil \frac{-\omega(Y_i + 1) + \sum_{j=i+1}^{n-1} T_{i,j} X_j + V_i}{T_{i,i}} \right\rceil .$$

In [4,5], a similar result can be obtained in dimension 2. However, the Hermite Normal Form formalization allows us to prove the result in higher dimension. To prove Theorem 6, let us first consider the following technical lemma:

Lemma 1. *Let $a, b, q, x \in \mathbb{Z}$ with $q > 0$, then*

$$a \leq qx < b \Leftrightarrow -\left\lceil \frac{-a}{q} \right\rceil \leq x < -\left\lceil \frac{-b}{q} \right\rceil .$$

Proof. The proof is detailed in Sect. A.6.

We can now prove the Theorem 6 (cf Sect. A.7). The implementation of the construction algorithm is straightforward: we just have to consider n nested loop such that the loop with level i goes from A_i to B_i quantities. See [6] for details in dimension 2 and 3.

4 A Generic QAT Algorithm

In Algorithm 1, we give the generic algorithm applying a contracting QAT f to an image $\mathcal{A}$ (see Fig. 3). The principle is that we give to each pixel Y of image $\mathcal{B}$ the average color of the paving P_Y in image $\mathcal{A}$.

If f is a dilating QAT, we obtain the very similar Algorithm 2 which principle is that firstly we replace f with f^{-1}, and then we give the color of each pixel Y of image $\mathcal{A}$ to each pixel of P_Y in image $\mathcal{B}$. In both algorithms, some elements cannot be computed in arbitrary dimension n. Indeed, even if there exist algorithms to compute the Hermite Normal Form of an arbitrary square integer matrix [7], there is no generic algorithm to obtain the minimal periodicities $\{\alpha_i\}$.

In the following sections, we detail the computation of the minimal periodicities in dimension 2 and 3.

Algorithm 1: Generic QAT algorithm for a contracting QAT

Input: a contracting QAT $f := (\omega, M, \boldsymbol{V})$, an image $\mathcal{A}: \mathbb{Z}^n \to \mathbb{Z}$
Output: a transformed image $\mathcal{B}: \mathbb{Z}^n \to \mathbb{Z}$
Compute the Hermite Normal Form of the matrix M;
Determine the minimal periodicities $\{\alpha_i\}$ and vectors $\{\boldsymbol{U}_i\}$;
Use Theorems 5 and 6 to compute the canonical pavings in the super-paving $\mathcal{P}$;
foreach $Y \in \mathcal{B}$ **do**
    Find Y^0 and W such that $P_Y = T_W P_{Y^0}$;
    foreach $Z \in P_{Y^0}$ **do**
        $c \leftarrow \mathcal{A}(T_W Z)$;   `// we read the color in the initial image`
        $sum \leftarrow sum + c$;
    $\mathcal{B}(Y) \leftarrow sum / |P_{Y^0}|$;   `// we set the color`

Algorithm 2: Generic QAT algorithm for a dilating QAT

Input: a dilating QAT $f := (\omega, M, \boldsymbol{V})$, an image $\mathcal{A}: \mathbb{Z}^n \to \mathbb{Z}$
Output: a transformed image $\mathcal{B}: \mathbb{Z}^n \to \mathbb{Z}$
Replace f with f^{-1};
Compute the Hermite Normal Form of the matrix M;
Determine the minimal periodicities $\{\alpha_i\}$ and vectors $\{\boldsymbol{U}_i\}$;
Use Theorems 5 and 6 to compute the canonical pavings in the super-paving $\mathcal{P}$;
foreach $Y \in \mathcal{A}$ **do**
    Find Y^0 and W such that $P_Y = T_W P_{Y^0}$;
    $c \leftarrow \mathcal{A}(Y)$;   `// we read the color in the initial image`
    foreach $Z \in P_{Y^0}$ **do**
        $\mathcal{B}(T_W Z) \leftarrow c$;   `// we set the color`

5 QAT in Dimension 2

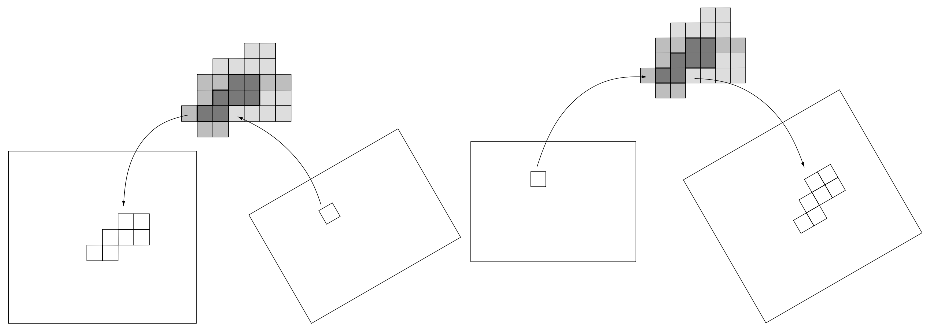
Let us consider the QAT (ω, M, V) with $M = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ and $V = \begin{pmatrix} e_0 \\ f_0 \end{pmatrix}$.

5.1 Hermite Normal Form and Paving Construction

Even if algorithms exist to compute Hermine Normal Form [7], we can define explicit formulas in dimension 2.

Lemma 2. *The Hermite Normal Form of M can be obtained such that $MH_1H_2 = T$ with:*

$$H_1 = \begin{cases} I_2 & \text{if } c_0 = 0, \\ \begin{pmatrix} d'_0 & u_0 \\ -c'_0 & v_0 \end{pmatrix} & \text{otherwise.} \end{cases} \qquad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = MH_1$$



(a) (b)

Fig. 3. Illustration in dimension 2 of the QAT algorithm when f is contracting (a) and dilating (b). In both cases, we use the canonical pavings contained in the super-paving to speed-up the transformation.

$$H_2 = \begin{cases} I_2 & \textit{if } a_1 > 0, \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \textit{otherwise} \end{cases}$$

with u_0 *and* v_0 *such that* $u_0 c_0 + v_0 d_0 = \gcd(c_0, d_0)$, *and* $c_0' = \frac{c_0}{\gcd(c_0,d_0)}$, $d_0' = \frac{d_0}{\gcd(c_0,d_0)}$.

Proof. To prove the lemma, we need to prove that $|det(H_1 H_2)| = 1$ and that $M H_1 H_2$ is an integer upper triangular matrix with $T_{ii} > 0$. Details are given in Sect. A.9.

In the following, we define $H = H_1 H_2$ and $MH = T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Thanks to Theorem 6, optimal algorithm can be designed to construct the paving $P_{i,j}$ associated to the pixel (i, j). First, using the notation of Theorem 6, we have:

$$A_1 = -\left\lceil \frac{-\omega j + f_0}{c} \right\rceil, B_1 = -\left\lceil \frac{-\omega(j+1) + f_0}{c} \right\rceil$$
$$A_0(y) = -\left\lceil \frac{-\omega i + e_0 + by}{a} \right\rceil, B_0(y) = -\left\lceil \frac{-\omega(i+1) + f_0 + by}{a} \right\rceil$$

Hence, Algorithm 3 details the $P_{i,j}$ construction. Note that the algorithm is optimal and output sensitive since it only scans $P_{i,j}$ points. Furthermore, this algorithm is very efficient since it only contains integer number computations without any *If* test. Note that similar algorithm exists in 2-D [4,5], but the Hermite Normal Form allows to have a compact algorithm which implementation is straightforward.

Algorithm 3: Paving Construction in 2-D

Input: a contracting QAT $f := (\omega, M, \boldsymbol{V})$ and the Hermite Normal Form $MH = T$
Output: paving $P_{i,j}$
$A_1 \leftarrow -\left[\frac{-\omega j + f_0}{c}\right]$;
$B_1 \leftarrow -\left[\frac{-\omega (j+1) + f_0}{c}\right]$;
for $y \leftarrow A_1$ **to** B_1 - *1* **do**
  $A_0 \leftarrow -\left[\frac{-\omega i + e_0 + by}{a}\right]$;
  $B_0 \leftarrow -\left[\frac{-\omega (i+1) + e_0 + by}{a}\right]$;
  for $x \leftarrow A_0$ **to** B_0 - *1* **do**
    $H \begin{pmatrix} x \\ y \end{pmatrix} \in P_{i,j}$;

5.2 Minimal Periodicity and Super-paving Computation

In this section, we first present explicit formulas to compute the minimal periods of a QAT. Let

$$a'_h = \frac{a}{\gcd(a, \omega)}, \omega'_h = \frac{\omega}{\gcd(a, \omega)}, Y = \begin{pmatrix} \omega'_h \\ 0 \end{pmatrix} .$$

The following theorem allows us to obtain the first periodicity along the $x-$axis.

Theorem 7 (Horizontal Periodicity). *Let $\alpha_h = a'_h$ and U be such that $U = HY$. Then,*

$$\alpha_h > 0 \quad P_{i+\alpha_h, j} \equiv P_{i,j} \quad \text{ and } \quad \forall (i,j) \in \mathbb{Z}^2, P_{i+\alpha_h, j} = T_U P_{i,j} .$$

Proof. The proof is detailed in Sec. A.10.

According to notations in Def. 8, we can conclude that $\alpha_h \in \mathcal{A}_0$. We now prove that α_h in the minimal period.

Theorem 8. *The period α_h is the minimal horizontal period,* i.e. $\alpha_h = \alpha_0$.

Proof. The proof is detailed in Sec. A.11.

Let us consider now the vertical periodicity. We define

$$c'_v = \frac{c}{\gcd(c, \omega)}, \omega'_v = \frac{\omega}{\gcd(c, \omega)}, a'_v = \frac{a}{\gcd(a, b\omega'_v, \omega)}, \phi = \frac{b\omega'_v}{\gcd(a, b\omega'_v, \omega)}, \omega''_v = \frac{\omega}{\gcd(a, b\omega'_v, \omega)},$$

$\alpha'_v = \gcd(a'_v, \omega''_v)$, u_1 and v_1 such that: $a'_v u_1 + \omega''_v v_1 = \gcd(a'_v, \omega''_v)(= \alpha'_v)$,

$$\beta_0 = -\phi v_1, Y = \begin{pmatrix} -\phi u_1 \\ \omega'_v \alpha'_v \end{pmatrix} .$$

Theorem 9 (Vertical Periodicity). *Let $\alpha_v = c'_v \alpha'_v$ and $U = HY$. Then,*

$$\alpha_v > 0 \quad P_{i, j+\alpha_v} \equiv P_{i+\beta_0, j} \quad \text{ and } \quad \forall (i,j) \in \mathbb{Z}^2, P_{i, j+\alpha_v} = T_U P_{i+\beta_0, j} .$$

Then, $\alpha_v \in \mathcal{A}_1$. Let show that α_v is the minimal vertical period :

Proof. The proof is given in Sec. A.12.

Theorem 10. *The period α_v is the minimal vertical period,* i.e. $\alpha_v = \alpha_1$.

Proof. The proof is detailed in Sec. A.13.

We now have both the horizontal and vertical periods and translation vectors to generate the pavings. We need now to specify the super-paving construction to compute the canonical tiles of the initial period. Using Theorem 5, we have

$$\theta_0 = \frac{\operatorname{lcm}(\alpha_0, \alpha_1)}{\alpha_0}, \theta_1 = \frac{\operatorname{lcm}(\alpha_0, \alpha_1)}{\alpha_1}$$

With both Theorem 5 and 6, we obtain

$$\mathcal{P} = \left\{ H \begin{pmatrix} x \\ y \end{pmatrix} / A'_1 \leq y < B'_1 \text{ and } A'_0(y) \leq x < B'_0(y) \right\}$$

where

$$A'_1 = -\left[\frac{\theta_1 f_0}{\theta_1 c}\right], B'_1 = -\left[\frac{-\omega \operatorname{lcm}(\alpha_0, \alpha_1) + \theta_1 f_0}{\theta_1 c}\right]$$

$$A'_0(y) = -\left[\frac{\theta_0 e_0 + \theta_0 b y}{\theta_0 a}\right], B'_0(y) = -\left[\frac{-\omega \operatorname{lcm}(\alpha_0, \alpha_1) + \theta_0 e_0 + \theta_0 b y}{\theta_0 a}\right]$$

Hence,

$$A'_1 = -\left[\frac{f_0}{c}\right], A'_0(y) = -\left[\frac{e_0 + by}{a}\right].$$

Furthermore, using the notations of Def. 8,

$$\frac{\omega \operatorname{lcm}(\alpha_0, \alpha_1)}{\theta_1 c} = \frac{\omega \alpha_1}{c} = \frac{\omega'_v \alpha_1}{c'_v} = \omega'_v \alpha'_v \in \mathbb{Z}, \quad \text{and} \quad \frac{\omega \operatorname{lcm}(\alpha_0, \alpha_1)}{\theta_0 a} = \frac{\omega \alpha_0}{a} = \frac{\omega'_h \alpha_0}{a'_h} = \omega'_h \in \mathbb{Z}$$

Finally,

$$B'_1 = \frac{\omega \operatorname{lcm}(\alpha_0, \alpha_1)}{\theta_1 c} - \left[\frac{f_0}{c}\right] = A'_1 + \frac{\omega \alpha_1}{c} \quad \text{and } B'_0(y) = \frac{\omega \operatorname{lcm}(\alpha_0, \alpha_1)}{\theta_0 a} - \left[\frac{e_0 + by}{a}\right] = A'_0(y) + \frac{\omega \alpha_0}{a}$$

For each point $X \in \mathcal{P}$, we need to determine the paving index Y to which X belongs to. Since $X \in P_Y \Leftrightarrow \left[\frac{MX+V}{\omega}\right] = Y$, we can design a simple algorithm (Algorithm 4) that construct all the initial period pavings while scanning points in $\mathcal{P}$. With initial period pavings and translation vectors, all other QAT pavings will be obtained using a simple translation. The computational cost of Alg. 4 exactly corresponds to the number of pavings in the initial period.

Proposition 2. *The number of pavings of the initial period is* $\omega'_v \omega''_v = \frac{\omega^2}{\gcd(c,\omega)\gcd(a, b\omega'_v, \omega)}$.

Proof. The proof is given in Section A.14.

Algorithm 4: Super-paving and initial period pavings construction in 2-D.

$A'_1 \leftarrow -\left[\frac{f_0}{c}\right]$;
for $y \leftarrow A'_1$ **to** $A'_1 + \frac{\omega\alpha_1}{c}$ *- 1* **do**
  $A'_0 \leftarrow -\left[\frac{e_0+by}{a}\right]$;
  for $x \leftarrow A'_0$ **to** $A'_0 + \frac{\omega\alpha_0}{a}$ *- 1* **do**
    $Y^0 \leftarrow \left[\frac{T\begin{pmatrix}x\\y\end{pmatrix}+V}{\omega}\right]$;
    $H\begin{pmatrix}x\\y\end{pmatrix} \in P_{Y^0}$;

5.3 QAT Algorithm in 2-D

In [6,8], we have sketched a set of generic algorithms to apply a QAT on an $n-$D image. To implement those algorithms in dimension 2, we first use Algorithm 4 to generate the initial period pavings. Then, to each point $Y = (i, j)$, we need to determine $Y^0 = (i^0, j^0)$ and W such that $P_Y = T_W P_{Y^0}$.

Using Theorem 4, $W = w_0U_0 + w_1U_1$ with

$$w_1 = \left[\frac{j}{\alpha_1}\right] \quad \text{and} \quad w_0 = \left[\frac{i + w_1\beta_0^1}{\alpha_0}\right], \text{and } (i^0, j^0) = \left(\left\{\frac{i + w_1\beta_0^1}{\alpha_0}\right\}, \left\{\frac{j}{\alpha_1}\right\}\right).$$

Algorithm 5 details the transformation algorithm. In this algorithm, $g(i, j)$ returns the image color at (i, j). Furthermore, $g(E)$ with $E \subset \mathbb{Z}^2$ returns a color associated to the set E (*e.g.* the mean color). Again, similar algorithm can be found in the literature in dimension 2 [3,4,5].

6 QAT in Dimension 3

In dimension 3, we use a similar framework as in 2-D: we first define the Hermite Normal Form, the minimal periods and then the transformation algorithm.

6.1 Hermite Normal Form and Paving Construction

Let us consider a QAT (ω, M, V) with $M = \begin{pmatrix}a_0 & b_0 & c_0\\ d_0 & e_0 & f_0\\ g_0 & h_0 & i_0\end{pmatrix}$ and $V = \begin{pmatrix}j_0\\k_0\\l_0\end{pmatrix}$.

In Sect. A.15, we present explicit formulas to compute the Hermite Normal Form in 3-D. In the following, we define $H = H_1H_2H_3H_4$ and $MH = T = \begin{pmatrix}a & b & c\\ 0 & d & e\\ 0 & 0 & f\end{pmatrix}$. Thanks Hermite decomposition, we have $a > 0$, $d > 0$ and $f > 0$. To construct the paving of index (i, j, k) and thanks to Theorem 6, we have:

Algorithm 5: QAT algorithm in 2-D

Input: A QAT (ω, M, V) and an image $g:\ [0, t_0] \times [0, t_1] \to \mathbb{Z}$
Output: an image $h:\ [min_0, max_0] \times [min_1, max_1] \to \mathbb{Z}$
Compute min_i and max_i quantities from t_i;
if f *dilating* **then**
    $f \leftarrow f^{-1}$;
    Compute the Hermite Normal Form of the matrix M;
    Compute the minimal periodicities $\{\alpha_0, \alpha_1\}$ and vectors $\{\boldsymbol{U}_0, \boldsymbol{U}_1\}$;
    Use Algorithm 4 to compute the canonical pavings in the super-paving $\mathcal{P}$;
    for $i \leftarrow 0$ **to** $t_0 - 1$ **do**
        for $j \leftarrow 0$ **to** $t_1 - 1$ **do**
            Compute W, i^0, j^0 ;
            $h(T_W P_{i^0, j^0}) \leftarrow g(i, j)$;
else
    Compute the Hermite Normal Form of the matrix M;
    Compute the minimal periodicities $\{\alpha_0, \alpha_1\}$ and vectors $\{\boldsymbol{U}_0, \boldsymbol{U}_1\}$;
    Use Algorithm 4 to compute the canonical pavings in the super-paving $\mathcal{P}$;
    for $i \leftarrow min_0$ **to** $max_0 - 1$ **do**
        for $j \leftarrow min_1$ **to** $max_1 - 1$ **do**
            Compute W, i^0, j^0 ;
            $h(i, j) \leftarrow g(T_W P_{i_0, j_0})$;

$$A_2 = -\left[\frac{-\omega k + l_0}{f}\right], B_2 = -\left[\frac{-\omega(k+1) + l_0}{f}\right]$$

$$A_1(z) = -\left[\frac{-\omega j + k_0 + ez}{d}\right], B_1(z) = -\left[\frac{-\omega(j+1) + k_0 + ez}{d}\right]$$

$$A_0(y, z) = -\left[\frac{-\omega i + j_0 + by + cz}{a}\right], B_0(y, z) = -\left[\frac{-\omega(i+1) + j_0 + by + cz}{a}\right]$$

Algorithm 6: Paving construction in 3-D

$A_2 \leftarrow -\left[\frac{-\omega k + l_0}{f}\right]$;
$B_2 \leftarrow -\left[\frac{-\omega(k+1) + l_0}{f}\right]$;
for $z \leftarrow A_2$ **to** B_2 - *1* **do**
    $A_1 \leftarrow -\left[\frac{-\omega j + k_0 + ez}{d}\right]$;
    $B_1 \leftarrow -\left[\frac{-\omega(j+1) + k_0 + ez}{d}\right]$;
    for $y \leftarrow A_1$ **to** B_1 - *1* **do**
        $A_0 \leftarrow -\left[\frac{-\omega i + j_0 + by + cz}{a}\right]$;
        $B_0 \leftarrow -\left[\frac{-\omega(i+1) + j_0 + by + cz}{a}\right]$;
        for $x \leftarrow A_0$ **to** B_0 - *1* **do**
            $H \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_{i,j,k}$;

6.2 Minimal Periodicity and Super-paving Construction

In dimension 3, we need to compute the periodicity along each dimension. Let us first denote $a'_h = \frac{a}{\gcd(a,\omega)}, \omega'_h = \frac{\omega}{\gcd(a,\omega)}, Y = \begin{pmatrix} \omega'_h \\ 0 \\ 0 \end{pmatrix}$

Theorem 11 (Horizontal Periodicity). *Let $\alpha_h = a'_h$ and $U = HY$. Then $\alpha_h > 0$, $P_{i+\alpha_h,j,k} \equiv P_{i,j,k}$ and $\forall (i,j,k) \in \mathbb{Z}^3, P_{i+\alpha_h,j,k} = T_U P_{i,j,k}$.*

Proof. The proof is detailed in Sect. A.16.

Theorem 12. *The period α_h is a minimal horizontal period,* i.e. $\alpha_h = \alpha_0$.

Proof. The proof is given in Sect. A.17

Concerning the vertical period, let:

$$d'_v = \frac{d}{\gcd(d,\omega)}, \omega'_v = \frac{\omega}{\gcd(d,\omega)}, a'_v = \frac{a}{\gcd(a, b\omega'_v, \omega)}, \phi = \frac{b\omega'_v}{\gcd(a, b\omega'_v, \omega)}, \omega''_v = \frac{\omega}{\gcd(a, b\omega'_v, \omega)},$$

$\alpha'_v = \gcd(a'_v, \omega''_v)$, u_1 and v_1 are such that : $a'_v u_1 + \omega''_v v_1 = \gcd(a'_v, \omega''_v)(= \alpha'_v)$,

$\beta_0 = -\phi v_1, Y = \begin{pmatrix} -\phi u_1 \\ \omega'_v \alpha'_v \\ 0 \end{pmatrix}$

Theorem 13 (Vertical Periodicity). *Let $\alpha_v = d'_v \alpha'_v, U = HY$. Then $\alpha_v > 0$, $P_{i,j+\alpha_v,k} \equiv P_{i+\beta_0,j,k}$ and $\forall (i,j,k) \in \mathbb{Z}^3, P_{i,j+\alpha_v,k} = T_U P_{i+\beta_0,j,k}$.*

Proof. The proof is given in Sect. A.18.

Theorem 14. *The period α_v is a minimal vertical period,* i.e. $\alpha_v = \alpha_1$.

Proof. The proof is detail-led in Sect. A.19.

And for the last period, let:

$$f'_d = \frac{f}{\gcd(\omega, f)}, \omega'_d = \frac{\omega}{\gcd(\omega, f)}, d'_d = \frac{d}{\gcd(d, e\omega'_d, \omega)}, \phi = \frac{e\omega'_d}{\gcd(d, e\omega'_d, \omega)}, \omega''_d = \frac{\omega}{\gcd(d, e\omega'_d, \omega)},$$

u_1 and v_1 are such that : $d'_d u_1 + \omega''_d v_1 = \gcd(d'_d, \omega''_d), \psi = c\omega'_d \gcd(d'_d, \omega''_d) - b\phi u_1,$

$$a'_d = \frac{a}{\gcd(a, \psi, \omega, \frac{\omega''_d b}{\gcd(d'_d, \omega''_d)})}, \psi' = \frac{\psi}{\gcd(a, \psi, \omega, \frac{\omega''_d b}{\gcd(d'_d, \omega''_d)})},$$

$$\omega'''_d = \frac{\omega}{\gcd(a, \psi, \omega, \frac{\omega''_d b}{\gcd(d'_d, \omega''_d)})}, \chi = \frac{\frac{\omega''_d b}{\gcd(d'_d, \omega''_d)}}{\gcd(a, \psi, \omega, \frac{\omega''_d b}{\gcd(d'_d, \omega''_d)})},$$

$\alpha''_d = \gcd(a'_d, \chi, \omega'''_d), \alpha'_d = \alpha''_d \gcd(d'_d, \omega''_d),$

u_2, v_2 and w_2 are such that : $a'_d u_2 + \chi v_2 + \omega'''_d w_2 = \gcd(a'_d, \chi, \omega'''_d)(= \alpha''_d),$

$$k = -\psi' v_2, \beta_0 = -\psi' w_2, \beta_1 = -\phi v_1 \alpha''_d - k\frac{d'_d}{\gcd(d'_d, \omega''_d)},$$

$$Y = \begin{pmatrix} -\psi' u_2 \\ -\phi u_1 \alpha''_d + k\frac{\omega''_d}{\gcd(d'_d, \omega''_d)} \\ \alpha'_d \omega'_d \end{pmatrix}$$

Theorem 15 (Depth Periodicity). *Let* $\alpha_d = \alpha'_d f'_d, U = HY$. *Then*

$$\alpha_d > 0 \quad P_{i,j,k+\alpha_d} \equiv P_{i+\beta_0,j+\beta_1,k} \qquad \text{and } \forall (i,j,k) \in \mathbb{Z}^3, P_{i,j,k+\alpha_d} = T_U P_{i+\beta_0,j+\beta_1,k}$$

Proof. The proof is detailed in Sect. A.20.

Theorem 16. *The period* α_d *is a minimal depth period,* i.e. $\alpha_d = \alpha_2$.

Proof. The proof is detailed in Sect. A.21.

Based on these periods, we can construct the super-paving and all the initial period pavings. As in dimension 2, for each point $X \in \mathcal{P}$, we need to determine the paving index Y to which X belongs to. Since $X \in P_Y \Leftrightarrow \left[\frac{MX+V}{\omega}\right] = Y$, Algorithm 7 details the initial period paving construction with scanning points in $\mathcal{P}$.

The computational cost of Alg. 7 exactly corresponds to the number of pavings in the initial period.

Proposition 3. *The number of pavings of the initial period is* $\omega'_d \omega''_d \omega'''_d$.

In the Proposition statement, we do not give the closed formula as in 2-D. However, $\omega'_d \omega''_d \omega'''_d$ is equal to ω^3 divided by a product of three gcd().

Proof. The proof is detailed in Sect. A.23.

Using Theorems 5 and 6, we have

$$\theta_0 = \frac{\operatorname{lcm}(\alpha_0, \alpha_1, \alpha_2)}{\alpha_0}, \theta_1 = \frac{\operatorname{lcm}(\alpha_0, \alpha_1, \alpha_2)}{\alpha_1}, \theta_1 = \frac{\operatorname{lcm}(\alpha_0, \alpha_1, \alpha_2)}{\alpha_2},$$

Algorithm 7: Super-paving and initial period pavings construction in 3-D.

$A'_2 \leftarrow -\left[\frac{l_0}{f}\right]$;
for $z \leftarrow A'_2$ **to** $A'_2 + \frac{\omega\alpha_2}{f}$ - *1* **do**
  $A'_1 \leftarrow -\left[\frac{k_0+ez}{d}\right]$;
  for $y \leftarrow A'_1$ **to** $A'_1 + \frac{\omega\alpha_1}{d}$ - *1* **do**
    $A'_0 \leftarrow -\left[\frac{j_0+by+cz}{a}\right]$;
    for $x \leftarrow A'_0$ **to** $A'_0 + \frac{\omega\alpha_0}{a}$ - *1* **do**
      $Y \leftarrow \left[\frac{T\begin{pmatrix}x\\y\\z\end{pmatrix}+V}{\omega}\right]$;
      $H\begin{pmatrix}x\\y\\z\end{pmatrix} \in P_Y$;

and

$$\mathcal{P} = \left\{ H\begin{pmatrix}x\\y\\z\end{pmatrix} / A'_2 \le z < B'_2, A'_1(z) \le y < B'_1(z) \text{ and } A'_0(y,z) \le x < B'_0(y,z) \right\},$$

with $A'_2 = -\left[\frac{l_0}{f}\right]$, $A'_1(z) = -\left[\frac{k_0+ez}{d}\right]$, $A'_0(y,z) = -\left[\frac{j_0+by+cz}{a}\right]$, $B'_2 = A'_2 + \frac{\omega\alpha_2}{f}$, $B'_1(z) = A'_1(z) + \frac{\omega\alpha_1}{d}$, and $B'_0(y,z) = A'_0(y,z) + \frac{\omega\alpha_0}{a}$ (see Sect. A.22 for details).

6.3 QAT Algorithm in 3-D

To obtain the overall QAT algorithm, we need to find both the initial period paving index and the translation vector associated to a given paving $P_{i,j,k}$. Hence, thanks to Theorem 4, we have

$$\forall (i,j,k) \in \mathbb{Z}^3, P_{i,j,k} = T_W P_{i^0,j^0,k^0} \text{ with } W = w_0U_0 + w_1U_1 + w_2U_2$$

and $0 \le k^0 = \left\{\frac{k}{\alpha_2}\right\} < \alpha_2,\ w_2 = \left[\frac{k}{\alpha_2}\right], \quad 0 \le j^0 = \left\{\frac{j + w_2\beta_1^2}{\alpha_1}\right\} < \alpha_1,\ w_1 = \left[\frac{j+w_2\beta_1^2}{\alpha_1}\right],$

$$0 \le i^0 = \left\{\frac{i + w_1\beta_0^1 + w_2\beta_0^2}{\alpha_0}\right\} < \alpha_0,\ w_0 = \left[\frac{i + w_1\beta_0^1 + w_2\beta_0^2}{\alpha_0}\right].$$

7 Experiments

The algorithms were implemented in both 2D and 3D, with different refinements in order to be able to compare the implementations. The `backward mapping` (`B. M.` for short) implementation let us compare the paving periodicity method with the widely used `backward mapping` method [9]. The `simple` implementation

Algorithm 8: QAT Algorithm in 3-D.

Input: A QAT (ω, M, V) and an image $g:\ [0,t_0]\times[0,t_1]\times[0,t_2]\to\mathbb{Z}$
Output: an image $h:\ [min_0, max_0]\times[min_1, max_1]\times[min_2, max_2]\to\mathbb{Z}$
Compute min_i and max_i quantities from t_i;
if f *dilating* **then**
  $f \leftarrow f^{-1}$;
  Compute the Hermite Normal Form of the matrix M;
  Compute the minimal periodicities $\{\alpha_0, \alpha_1, \alpha_2\}$ and vectors $\{\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2\}$;
  Use Algorithm 7 to compute the canonical pavings in the super-paving $\mathcal{P}$;
  for $i \leftarrow 0$ **to** $t_0 - 1$ **do**
    for $j \leftarrow 0$ **to** $t_1 - 1$ **do**
      for $k \leftarrow 0$ **to** $t_2 - 1$ **do**
        Compute W, i^0, j^0, k^0 ;
        $h(T_W P_{i^0,j^0,k^0}) \leftarrow g(i,j,k)$;
else
  Compute the Hermite Normal Form of the matrix M;
  Compute the minimal periodicities $\{\alpha_0, \alpha_1, \alpha_2\}$ and vectors $\{\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2\}$;
  Use Algorithm 7 to compute the canonical pavings in the super-paving $\mathcal{P}$;
  for $i \leftarrow min_0$ **to** $max_0 - 1$ **do**
    for $j \leftarrow min_1$ **to** $max_1 - 1$ **do**
      for $k \leftarrow 0$ **to** $t_2 - 1$ **do**
        Compute W, i^0, j^0, k^0 ;
        $h(i,j,k) \leftarrow g(T_W P_{i^0,j^0,k^0})$;

does not use pavings periodicity and uses algorithms 3 and 6 for every paving. The `periodicity` implementation uses the periodicity and the algorithms 5 and 8. The `noMultiply` implementation additionally uses a method presented in [4] which uses a handling of remains instead of computing a matrix product in 4 and 7. The experiments are performed on an Intel© Centrino© Duo T2080 (2 x 1.73 GHz) in monothread and we give on one hand the time of computation and on the other hand the number of elementary instructions. The QATs used are the following : In 2-D:

$$\left(\omega, \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \text{ where } \omega = \begin{cases} 10 \text{ for the contracting application} \\ 5 \text{ for the isometry} \\ 2 \text{ for the dilating application} \end{cases}$$

In 3-D:

$$\left(\omega, \begin{pmatrix} 9 & -20 & -12 \\ 12 & 15 & -16 \\ 20 & 0 & 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) \text{ where } \omega = \begin{cases} 100 \text{ for the contracting application} \\ 25 \text{ for the isometry} \\ 4 \text{ for the dilating application} \end{cases}$$

The pictures are of size : 200 x 171 in 2-D and 10 x 10 x 10 in 3-D (simple cube).

Figure 4 illustrates the results in dimension 2. As expected, when comparing `B.M.` and `Periodicity`, results are similar for both contracting and isometry QATs. Differences appear when we dilating QAT is considered. Indeed, since a unique color is associated to a paving in the `Periodicity` algorithm, the transformed image contains sharp edges (Fig 4-(l)) On the other hand, the interpolation process in the `B.M.` algorithm makes the image blurred. To compare the time efficiency (Table 2), we have considered two quantities: the total number of elementary operations of the main loop[3] and the overall computational time in seconds.

	2D - instructions (time in sec.)			
	`B.M.`	`simple`	`Periodicity`	`noMultiply`
contracting	1 607 774 (0.036)	64 536 315 (0.06)	29 578 702 (0.036)	27 679 044 (0.036)
isometry	63 058 160 (0.112)	57 619 374 (0.064)	39 682 795 (0.056)	35 875 892 (0.044)
dilating	391 622 017 (0.404)	185 956 768 (0.12)	87 490 567 (0.084)	83 472 387 (0.078)

	3D - instructions (time in sec.)		
	`B.M.`	`simple`	`Periodicity`
contracting	15 864 982 (0.02)	47 303 861 (0.052)	12 865 125 (0.012)
isometry	750 102 224 (0.416)	51 121 827 (0.068)	15 234 007 (0.016)
dilating	170 072 035 547 (79.637)	2 479 676 409 (1.384)	7 760 893 011 (0.632)

Table 2. Comparative evaluation in 2-D and 3-D.

Table 2 and Figure 5 present the result in dimension 3. For the sake of clarity, we have only considered an input binary image but the transformation algorithms can be applied to 3-D color images.

8 Conclusion and Future Works

In this paper, we have demonstrated that in higher dimension, Quasi-Affine Transformations contain arithmetical properties leading to the fact that the induced pavings are $n-$periodic. Furthermore, thanks to the Hermite Normal Form of the QAT matrix, we have presented efficient algorithms to construct a given paving and to compute a set of canonical pavings. From all these theoretical results, fast transformation algorithms have been designed which outperform classical ones.

However, several future works exist. First, as detailed in Sections 3.1 and 3.3, the super-paving of a QAT contains a set of arithmetically distinct pavings. However, two arithmetical distinct pavings may be geometrically equivalent. Hence, a subset of the super-paving may be enough to design a fast algorithm. In dimension 2, in [1,3,4,5], the authors have investigated another structure, so-called generative strip, which removes some arithmetical distinct pavings whose geometry are identical. Even if the generalization in higher dimension of this object is not trivial, it may be interesting to investigate theoretical techniques to reduce the canonical paving set. Finally, a generic algorithm to compute the minimal periodicities is challenging.

[3] obtained with the `valgrind` and profiling tool.

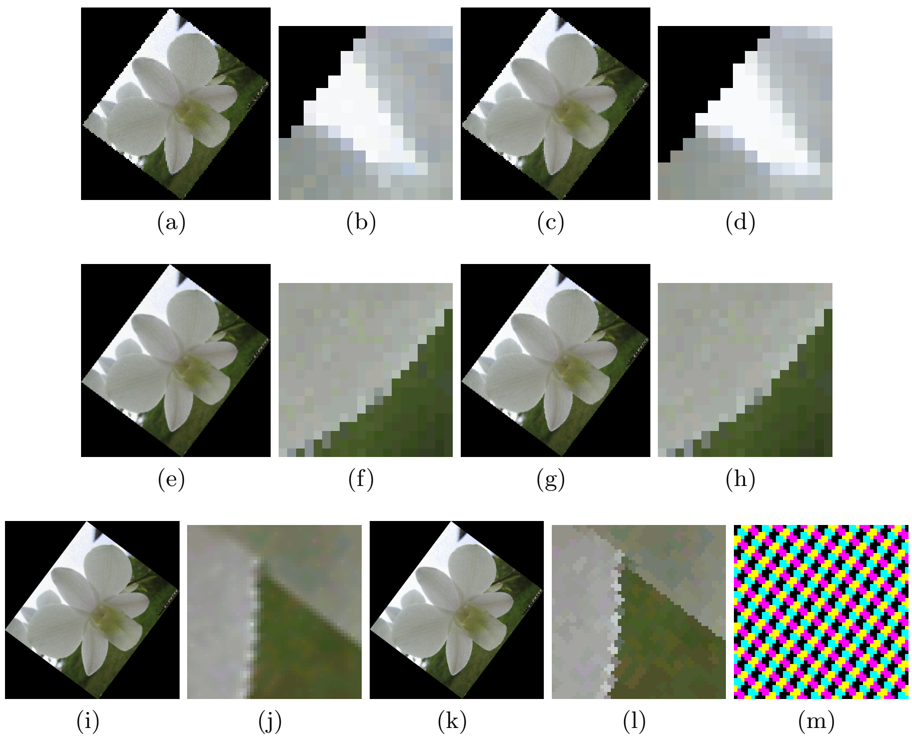
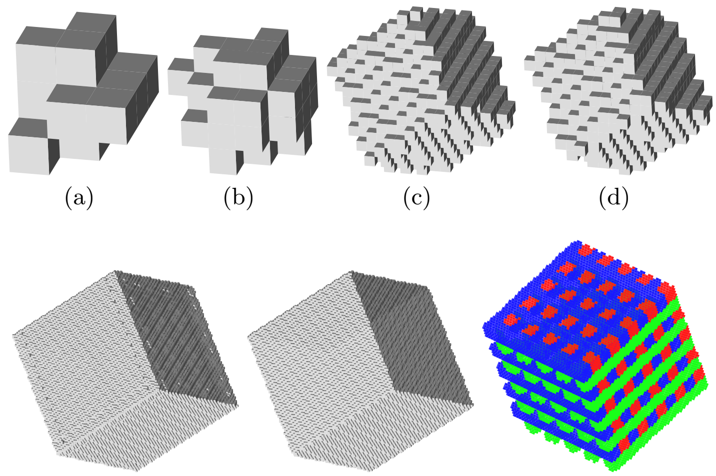


Fig. 4. Results in dimension 2: $(a-d)$ Contracting QAT (`B.M.` $(a-b)$ and `Periodicity` $(c-d)$); $(e-h)$ Isometry (`B.M.` and `Periodicity`); $(i-l)$ Dilating (`B.M.` and `Periodicity`). (m) and (n) illustrate the paving structure of the dilating QAT.



(a) (b) (c) (d)

(e) (f) (g)

Fig. 5. Results in dimension 3: $(a-b)$ Contracting (`B.M.` and `Periodicity`), $(c-d)$ Isometry and $(e-f)$ Dilating (`B.M.` and `Periodicity`). (g) illustrates the paving structure of the dilating QAT.

A Proofs

A.1 Theorem 1

Proof. Let P_Y be a paving arithmetically equivalent to P_Z, let us show that P_Z is arithmetically equivalent to P_Y. Let $X' \in P_Z$, let $X_0 \in P_Y$, there is $X_0' \in P_Z$ such that:

$$\left\{\frac{MX_0 + V}{\omega}\right\} = \left\{\frac{MX_0' + V}{\omega}\right\} \tag{9}$$

Let $X = X_0 + X' - X_0'$, thus

$$\begin{aligned} MX + V &= (MX_0 + V) + (MX' + V) - (MX_0' + V) \\ &= (\omega Y + \left\{\frac{MX_0 + V}{\omega}\right\}) + (\omega Z + \left\{\frac{MX' + V}{\omega}\right\}) \\ &\qquad - (\omega Z + \left\{\frac{MX_0' + V}{\omega}\right\}) \\ &= \omega \left\lfloor\frac{MX_0 + V}{\omega}\right\rfloor + \left\{\frac{MX' + V}{\omega}\right\} \end{aligned}$$

Hence, from the uniqueness of the Euclidean positive remainder division,

$$\left\lfloor\frac{MX + V}{\omega}\right\rfloor = \left\lfloor\frac{MX_0 + V}{\omega}\right\rfloor \text{ hence } X \in P_Y \text{ and } \left\{\frac{MX + V}{\omega}\right\} = \left\{\frac{MX' + V}{\omega}\right\}$$

□

A.2 Theorem 2

Proof. Let us suppose that $P_Y \equiv P_Z$, and let $X_0 \in P_Y$ and $X_0' \in P_Z$ be such that :

$$\left\{\frac{MX_0 + V}{\omega}\right\} = \left\{\frac{MX_0' + V}{\omega}\right\}$$

Let $\boldsymbol{v} = X_0 - X_0'$. To prove the theorem, we need to prove that $P_Y = T_{\boldsymbol{v}} P_Z$. We first prove that $P_Y \subset T_{\boldsymbol{v}} P_Z$. If $X \in P_Y$,

$$\begin{aligned} M(X - \boldsymbol{v}) + V &= M(X + X_0' - X_0) + V \\ &= (MX + V) + (MX_0' + V) - (MX_0 + V) \\ &= (\omega Y + \left\{\frac{MX + V}{\omega}\right\}) + (\omega Z + \left\{\frac{MX_0' + V}{\omega}\right\}) \\ &\qquad - (\omega Y + \left\{\frac{MX_0 + V}{\omega}\right\}) \\ &= \omega Z + \left\{\frac{MX + V}{\omega}\right\} \end{aligned}$$

Therefore $X - \boldsymbol{v} \in P_Z$, and thus $X \in T_{\boldsymbol{v}} P_Z$. We now prove that $P_Y \supset T_{\boldsymbol{v}} P_Z$. If $X' \in P_Z$,

$$\begin{aligned}
M(X' + \boldsymbol{v}) + V &= M(X' + X_0 - X_0') + V \\
&= (MX' + V) + (MX_0 + V) - (MX_0' + V) \\
&= (\omega Z + \left\{\frac{MX' + V}{\omega}\right\}) + (\omega Y + \left\{\frac{MX_0 + V}{\omega}\right\}) \\
&\qquad - (\omega Z + \left\{\frac{MX_0' + V}{\omega}\right\}) \\
&= \omega Y + \left\{\frac{MX' + V}{\omega}\right\}
\end{aligned}$$

Therefore $X' + \boldsymbol{v} \in P_Y$.□

A.3 Theorem 3

Proof. Given $0 \leq i < n$, let us suppose that $\forall 0 \leq j < i, \beta_j = 0$ and $\alpha = |\det(M)|$. Let $Y \in \mathbb{Z}^n$, $X \in P_Y$, and

$$X' = X + \frac{\det(M)}{|\det(M)|} \omega \operatorname{com}(M)^t e_i$$

with e_i being the $i-$th vector of the canonical basis of $\mathbb{R}^n$. We prove that $\mathcal{A}_i \neq \emptyset$ since $P_Y \equiv P_{Y + \alpha e_i}$:

$$\begin{aligned}
MX' + V &= MX + V + M \frac{\det(M)}{|\det(M)|} \omega \operatorname{com}(M)^t e_i \\
&= \omega Y + \left\{\frac{MX + V}{\omega}\right\} + \omega \frac{\det(M)}{|\det(M)|} M \operatorname{com}(M)^t e_i \\
&= \omega Y + \left\{\frac{MX + V}{\omega}\right\} + \omega \frac{\det(M)}{|\det(M)|} det(M) e_i \\
&= \omega Y + \left\{\frac{MX + V}{\omega}\right\} + \omega |\det(M)| e_i \\
&= \omega(Y + \alpha e_i) + \left\{\frac{MX + V}{\omega}\right\}
\end{aligned}$$

Hence, $\left\{\frac{MX'+V}{\omega}\right\} = \left\{\frac{MX+V}{\omega}\right\}$ and thus $X' \in P_{Y+\alpha e_i}$. Finally, $P_{Y+\alpha e_i} \equiv P_Y$ which proves that $\alpha \in \mathcal{A}_i$.□

A.4 Theorem 4

Proof. Let us denote $\mathcal{T}(j)$ the proposition

$$P_{y_0,\ldots,y_{n-1}} = T_{\sum_{i=j}^{n-1} w_i U_i} P_{y_0 + \sum_{i=j}^{n-1} w_i \beta_0^i, \ldots, y_{j-1} + \sum_{i=j}^{n-1} w_i \beta_{j-1}^i, y_j^0, \ldots, y_{n-1}^0} \,.$$

We consider the following induction: given $n > p \geq 0$, we suppose $\mathcal{T}(p+1)$ and prove $\mathcal{T}(p)$. As a consequence of Def. 8,

$$\forall(z_1, \ldots, z_{n-1}) \in \mathbb{Z}^n, P_{z_0,\ldots,z_p+\alpha_p,\ldots,z_{n-1}} = T_{U_p} P_{z_0+\beta_0^p,\ldots,z_{p-1}+\beta_{p-1}^p,z_p,\ldots,z_{n-1}} \, .$$

Hence, $\forall k \in \mathbb{Z}$, we have

$$P_{z_0,\ldots,z_p+k\alpha_p,\ldots,z_{n-1}} = T_{kU_p} P_{z_0+k\beta_0^p,\ldots,z_{p-1}+k\beta_{p-1}^p,z_p,\ldots,z_{n-1}} \, .$$

With

$$\begin{cases} k = w_p \\ \forall 0 \leq j < p, z_j = y_j + \sum_{i=p+1}^{n-1} w_i \beta_j^i \\ \forall p \leq j < n, z_j = y_j^0 \end{cases}$$

we obtain

$$\begin{aligned} &P_{y_0+\sum_{i=p+1}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p+1}^{n-1} w_i\beta_{p-1}^i,y_p^0+w_p\alpha_p,y_{p+1}^0,\ldots,y_{n-1}^0} \\ &\qquad = T_{w_pU_p} P_{y_0+\sum_{i=p}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p}^{n-1} w_i\beta_{p-1}^i,y_p^0,\ldots,y_{n-1}^0} \, . \end{aligned} \tag{10}$$

Since $\mathcal{T}(p+1)$ is true, and since

$$y_p + \sum_{j=p+1}^{n-1} w_j \beta_p^j = y_p^0 + \alpha_p w_p \, ,$$

we have

$$P_{y_0,\ldots,y_{n-1}} = T_{\sum_{i=p+1}^{n-1} w_i U_i} P_{y_0+\sum_{i=p+1}^{n-1} w_i\beta_0^i,\ldots,y_{p-1}+\sum_{i=p+1}^{n-1} w_i\beta_{p-1}^i,y_p^0+\alpha_p w_p,y_{p+1}^0,\ldots,y_{n-1}^0} \tag{11}$$

We can identify the left side of Eq. (13) to the right part of the right side of (11), summing up the translation vectors leads to $\mathcal{T}(p)$.

Since $\mathcal{T}(n) : P_{y_0,\ldots,y_{n-1}} = T_0 P_{y_0,\ldots,y_{n-1}}$ is trivial, we prove $\mathcal{T}(0)$ and thus the theorem.□

A.5 Theorem 5

Proof. Let $X \in \mathbb{Z}^n$,

$$\begin{aligned} X \in \mathcal{P} &\Leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \left[\frac{MX+V}{\omega}\right] < \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leq \frac{MX+V}{\omega} < \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \text{ since } \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z} \end{aligned}$$

$$\Leftrightarrow \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \leq \begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} \frac{MX+V}{\omega} < \begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} \begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_{n-1}\end{pmatrix}$$

since $\theta_0, \ldots, \theta_{n-1} > 0$

$$\Leftrightarrow \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \leq \begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} \frac{MX+V}{\omega} < \mathrm{lcm}_{0\leq i<n}(\alpha_i) \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \leq \frac{\begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} MX + \begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} V}{\omega\, \mathrm{lcm}_{0\leq i<n}(\alpha_i)} < \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$$\Leftrightarrow \left[\frac{\begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} MX + \begin{pmatrix}\theta_0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \theta_{n-1}\end{pmatrix} V}{\omega\, \mathrm{lcm}_{0\leq i<n}(\alpha_i)}\right] = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

This ends the proof since the last statement prove that X belongs to the paving $(0, \ldots, 0)$ of the QAT defined in Eq. (7).□

A.6 Lemma 1

Proof.

$$\begin{aligned} -a = \left[\frac{-a}{q}\right] q + \left\{\frac{-a}{q}\right\} \text{ with } 0 \leq \left\{\frac{-a}{q}\right\} < |q| = q \Rightarrow -\left[\frac{-a}{q}\right] &= \frac{a + \left\{\frac{-a}{q}\right\}}{q} \\ \Rightarrow -\left[\frac{-a}{q}\right] &< \frac{a}{q} + 1 \\ \Rightarrow -\left[\frac{-a}{q}\right] - \frac{a}{q} &< 1 \end{aligned}$$

Hence $\frac{a}{q} \leq x \Leftrightarrow -\left[\frac{-a}{q}\right] < x+1 \Leftrightarrow -\left[\frac{-a}{q}\right] \leq x$ (since x and $-\left[\frac{-a}{q}\right] \in \mathbb{Z}$)

$$
\begin{aligned}
-b = \left\lfloor \frac{-b}{q} \right\rfloor q + \left\{ \frac{-b}{q} \right\} \text{ with } 0 \leq \left\{ \frac{-b}{q} \right\} < |q| = q \Rightarrow - \left\lfloor \frac{-b}{q} \right\rfloor &= \frac{b + \left\{ \frac{-b}{q} \right\}}{q} \\
\Rightarrow \frac{b}{q} &\leq - \left\lfloor \frac{-b}{q} \right\rfloor \\
\Rightarrow 0 &\leq - \left\lfloor \frac{-b}{q} \right\rfloor - \frac{b}{q}
\end{aligned}
$$

$$
\text{Hence } x < \frac{b}{q} \Leftrightarrow x < - \left\lfloor \frac{-b}{q} \right\rfloor
$$

$$
\begin{aligned}
\text{Finally } a \leq qx < b &\Leftrightarrow \frac{a}{q} \leq x < \frac{b}{q} \\
&\Leftrightarrow - \left\lfloor \frac{-a}{q} \right\rfloor \leq x < - \left\lfloor \frac{-b}{q} \right\rfloor . \square
\end{aligned}
$$

A.7 Theorem 6

Proof. Let $X, Y, Z \in \mathbb{Z}^n$ such that $X = H^{-1} Z$,

$$
\begin{aligned}
Z \in P_Y &\Leftrightarrow \left\lfloor \frac{MZ + V}{\omega} \right\rfloor = Y \\
&\Leftrightarrow \left\lfloor \frac{TX + V}{\omega} \right\rfloor = Y \\
&\Leftrightarrow \forall 0 \leq i < n, \omega y_i \leq \sum_{j=i}^{n-1} T_{i,j} X_j + V_i < \omega (y_i + 1) \\
&\Leftrightarrow \forall 0 \leq i < n, \omega y_i - \sum_{j=i+1}^{n-1} T_{i,j} X_j - V_i \leq T_{i,i} X_i < \omega (y_i + 1) - \sum_{j=i+1}^{n-1} T_{i,j} X_j - V_i \,.
\end{aligned}
$$

Thanks to Lemma 1, $Z \in P_Y$ is equivalent to

$$
\forall 0 \leq i < n, A_i(X_{i+1}, \ldots, X_{n-1}) \leq X_i < B_i(X_{i+1}, \ldots, X_{n-1}) . \square
$$

A.8 Proposition 1

Proof. Let $(a, b, c) \in \mathbb{Z}^3$, common divisors to a, b and c divide a and b, and therefore divide $\gcd(a, b)$. Conversely, common divisors to $\gcd(a, b)$ and c also divide a and b. Therefore, common divisors to a, b and c are exactly common divisors to $\gcd(a, b)$ and c. Hence:

$$
\gcd(a, b, c) = \gcd(\gcd(a, b), c)
$$

Bezout's identity gives

$$
\begin{aligned}
&\exists (u_0, v_0) \in \mathbb{Z}^2 / a u_0 + b v_0 = \gcd(a, b) \\
\text{and } &\exists (u_1, v_1) \in \mathbb{Z}^2 / \gcd(a, b) u_1 + c v_1 = \gcd(\gcd(a, b), c) = \gcd(a, b, c)
\end{aligned}
$$

Hence

$$au_0u_1 + bv_0u_1 + cv_1 = \gcd(a, b, c)$$

And the result is obtained with $u = u_0u_1$, $v = v_0u_1$ and $w = v_1$.□

A.9 Lemma 2

Proof. First of all, let us define the H_1 matrix. If $c_0 = 0$, let us define $H_1 = I_2$[4]. Otherwise, let u_0 and v_0 such that:

$$u_0c_0 + v_0d_0 = \gcd(c_0, d_0)\,,$$

and $c_0' = \frac{c_0}{\gcd(c_0,d_0)}$, $d_0' = \frac{d_0}{\gcd(c_0,d_0)}$.

Let $H_1 = \begin{pmatrix} d_0' & u_0 \\ -c_0' & v_0 \end{pmatrix}$.

Hence we have $\det(H_1) = u_0c_0' + v_0d_0' = 1$ and $MH_1 = \begin{pmatrix} a_0d_0' - b_0c_0' & a_0u_0 + b_0v_0 \\ c_0d_0' - d_0c_0' = 0 & c_0u_0 + d_0v_0 \end{pmatrix}$

In both cases, we have $|\det(H_1)| = 1$ and MH_1 us upper triangular. Let us denote

$$MH_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}.$$

In order to ensure that diagonal coefficients are positive, we can first observe that

$$c_1 = c_0u_0 + d_0v_0 = \gcd(c_0, d_0) > 0\,.$$

Hence, if $a_1 > 0$, we define $H_2 = I_2$. Otherwise, we define H_2 such that

$$H_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finally, in both cases, we have $|\det(H_2)| = 1$ and MH_1H_2 is upper triangular with positive integers in the diagonal. To conclude, we denote $H = H_1H_2$ and $MH = T$, leading to the Hermite Normal Form.□

A.10 Theorem 7

Proof. First, since $a > 0$ (Hermite Normal Form), we have $\alpha_h > 0$. Then, $MU = TY = \begin{pmatrix} a\omega_h' \\ 0 \end{pmatrix} = \begin{pmatrix} a_h'\omega \\ 0 \end{pmatrix} = \omega \begin{pmatrix} \alpha_h \\ 0 \end{pmatrix}$. Given $X \in \mathbb{Z}^2$,

$$\begin{aligned} M(X-U)+V &= MX + V - MU \\ &= \omega\left[\frac{MX+V}{\omega}\right] + \left\{\frac{MX+V}{\omega}\right\} - \omega\begin{pmatrix} \alpha_h \\ 0 \end{pmatrix} \\ &= \omega\left(\left[\frac{MX+V}{\omega}\right] - \begin{pmatrix} \alpha_h \\ 0 \end{pmatrix}\right) + \left\{\frac{MX+V}{\omega}\right\} \end{aligned}$$

[4] I_n denotes the identity matrix in $M_n(\mathbb{Z})$.

Hence, $\left[\frac{M(X-U)+V}{\omega}\right] = \left[\frac{MX+V}{\omega}\right] - \begin{pmatrix}\alpha_h\\0\end{pmatrix}$ and $\left\{\frac{M(X-U)+V}{\omega}\right\} = \left\{\frac{MX+V}{\omega}\right\}$.

To conclude, let $(i,j) \in \mathbb{Z}^2$, if $X \in P_{i+\alpha_h,j}$, we have $X - U \in P_{i,j}$ and therefore $P_{i+\alpha_h,j} \equiv P_{i,j}$. This also gives the vector of translation : $X-(X-U) = U$ $\square$

A.11 Theorem 8

Proof. Let us first prove that α_h divides α_0. Using Def. 8, we have

$$\forall (i,j) \in \mathbb{Z}^2, P_{i+\alpha_0,j} = T_U P_{i,j}$$

Given $(i,j) \in \mathbb{Z}^2$, $X \in P_{i+\alpha_0,j}$ and $X' = X - U$, we have $X' \in P_{i,j}$ and $\left\{\frac{MX+V}{\omega}\right\} = \left\{\frac{MX'+V}{\omega}\right\}$. If we denote $H^{-1}U = \begin{pmatrix}x\\y\end{pmatrix}$, we obtain

$$\begin{aligned} T\begin{pmatrix}x\\y\end{pmatrix} &= MU = M(X-X') \\ &= (MX+V)-(MX'+V) \\ &= \omega\begin{pmatrix}i+\alpha_0\\j\end{pmatrix} - \omega\begin{pmatrix}i\\j\end{pmatrix} = \omega\begin{pmatrix}\alpha_0\\0\end{pmatrix}\,, \end{aligned}$$

which implies

$$\begin{cases} ax+by = \omega\alpha_0 \\ cy = 0 \end{cases} . \tag{12}$$

Thanks to the Hermite Normal Form, $c \neq 0$, hence we have $y = 0$. Then,

$$(12) \Rightarrow ax = \omega\alpha_0 \Rightarrow a'_h x = \omega'_h\alpha_0 \Rightarrow a'_h|\omega'_h\alpha_0,\ \text{or}\ \gcd(a'_h,\omega'_h) = 1\,.$$

Thanks to the Gauss Theorem, $a'_h|\alpha_0$ and thus $\alpha_h|\alpha_0$. Finally, by definition of α_0, $\alpha_h = \alpha_0$.$\square$

A.12 Theorem 9

Proof. Since $c > 0$ (Hermite Normal Form), we have $\alpha_v > 0$. First,

$$\begin{aligned} MU = TY &= \begin{pmatrix}-a\phi u_1 + b\omega'_v\alpha'_v\\ c\omega'_v\alpha'_v\end{pmatrix} \\ &= \begin{pmatrix}-a'_v b\omega'_v u_1 + b\omega'_v\alpha'_v\\ c'_v\omega\alpha'_v\end{pmatrix} = \begin{pmatrix}b\omega'_v(-a'_v u_1 + \alpha'_v)\\ \omega\alpha_v\end{pmatrix} \\ &= \begin{pmatrix}b\omega'_v\omega''_v v_1\\ \omega\alpha_v\end{pmatrix} = \begin{pmatrix}\phi\omega v_1\\ \omega\alpha_v\end{pmatrix} \\ &= \omega\begin{pmatrix}-\beta_0\\ \alpha_v\end{pmatrix} \end{aligned}$$

Let $X \in \mathbb{Z}^2$,

$$\begin{aligned}
M(X-U)+V &= MX+V-MU \\
&= \omega\left[\frac{MX+V}{\omega}\right]+\left\{\frac{MX+V}{\omega}\right\}-\omega\begin{pmatrix}-\beta_0\\ \alpha_v\end{pmatrix} \\
&= \omega\left(\left[\frac{MX+V}{\omega}\right]-\begin{pmatrix}-\beta_0\\ \alpha_v r\end{pmatrix}\right)+\left\{\frac{MX+V}{\omega}\right\}.
\end{aligned}$$

Hence, $\left[\frac{M(X-U)+V}{\omega}\right]=\left[\frac{MX+V}{\omega}\right]-\begin{pmatrix}-\beta_0\\ \alpha_v\end{pmatrix}$ and $\left\{\frac{M(X-U)+V}{\omega}\right\}=\left\{\frac{MX+V}{\omega}\right\}$.

Let $(i,j)\in\mathbb{Z}^2$, if $X \in P_{i,j+\alpha_v}$, we have $X-U \in P_{i+\beta_0,j}$ and therefore $P_{i,j+\alpha_v} \equiv P_{i+\beta_0,j}$. This also gives the vector of translation : $X-(X-U)=U$
□

A.13 Theorem 10

Proof. The number of pavings of the initial period is equal to the number of points in the super-paving. Hence,

$$\begin{aligned}
\frac{\omega\alpha_0}{a}\frac{\omega\alpha_1}{c} &= \frac{\omega^2\alpha_0\alpha_1}{ac} \\
&= \omega^2\frac{\frac{a}{\gcd(a,\omega)}\frac{c\gcd(a'_v,\omega''_v)}{\gcd(c,\omega)}}{ac} \\
&= \omega^2\frac{\gcd(a'_v,\omega''_v)}{\gcd(a,\omega)\gcd(c,\omega)} \\
&= \omega^2\frac{\gcd(\frac{a}{\gcd(a,b\omega'_v,\omega)},\frac{\omega}{\gcd(a,b\omega'_v,\omega)})}{\gcd(a,\omega)\gcd(c,\omega)} \\
&= \omega^2\frac{\gcd(a,\omega)}{\gcd(a,\omega)\gcd(c,\omega)\gcd(a,b\omega'_v,\omega)} \\
&= \frac{\omega}{\gcd(c,\omega)}\frac{\omega}{\gcd(a,b\omega'_v,\omega)} \\
&= \omega'_v\omega''_v
\end{aligned}$$

□

A.14 Proposition 2

Proof. The number of pavings of the initial period is equal to the number of points in the super-paving. Hence,

$$\begin{aligned}
\frac{\omega\alpha_0}{a}\frac{\omega\alpha_1}{d}\frac{\omega\alpha_2}{f} &= \frac{\omega}{\gcd(a,\omega)}\frac{\omega\gcd(a'_v,\omega''_v)}{\gcd(d,\omega)}\frac{\omega\gcd(a'_d,\chi,\omega'''_d)\gcd(d'_d,\omega''_d)}{\gcd(f,\omega)} \\
&= \frac{\omega}{\gcd(a,\omega)}\frac{\omega\gcd(a,\omega)}{\gcd(d,\omega)\gcd(a,b\omega'_v,\omega)}\frac{\omega\gcd(a'_d,\chi,\omega'''_d)\gcd(d,\omega)}{\gcd(f,\omega)\gcd(d,e\omega'_d,\omega)}
\end{aligned}$$

$$
\begin{aligned}
&= \omega \frac{\omega}{\gcd(a, b\omega'_v, \omega)} \frac{\omega \gcd(a'_d, \chi, \omega'''_d)}{\gcd(f, \omega)\gcd(d, e\omega'_d, \omega)} \\
&= \omega'_d \omega''_d \frac{\omega \gcd(a'_d, \chi, \omega'''_d)}{\gcd(a, b\omega'_v, \omega)}
\end{aligned}
$$

$$
\begin{aligned}
&= \omega'_d \omega''_d \frac{\omega \gcd(a, \frac{b\omega''_d}{\gcd(d'_d, \omega''_d)}, \omega)}{\gcd(a, b\frac{\omega}{\gcd(d,\omega)}, \omega)\gcd(a, \psi, \omega, \frac{b\omega''_d}{\gcd(d'_d, \omega''_d)})} \\
&= \omega'_d \omega''_d \frac{\omega \gcd(a, \frac{b\omega}{\gcd(d,\omega)}, \omega)}{\gcd(a, b\frac{\omega}{\gcd(d,\omega)}, \omega)\gcd(a, \psi, \omega, \frac{b\omega''_d}{\gcd(d'_d, \omega''_d)})} \\
&= \omega'_d \omega''_d \frac{\omega}{\gcd(a, \psi, \omega, \frac{b\omega''_d}{\gcd(d'_d, \omega''_d)})} \\
&= \omega'_d \omega''_d \omega'''_d
\end{aligned}
$$

□

A.15 Lemma 3

Lemma 3. *The Hermite Normal Form of M is such that $MH_1H_2H_3H_4 = T$ with:*

$$
H_1 = \begin{cases} I_3 & \text{if } g_0 = 0, \\ \begin{pmatrix} h'_0 & u_0 & 0 \\ -g'_0 & v_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases} \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} = MH_1
$$

$$
H_2 = \begin{cases} I_3 & \text{if } g_1 = 0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & h'_1 & u_1 \\ 0 & -g'_1 & v_1 \end{pmatrix} & \text{otherwise.} \end{cases} \qquad \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} = MH_1H_2
$$

$$
H_3 = \begin{cases} I_3 & \text{if } d_2 = 0, \\ \begin{pmatrix} e'_2 & p_2 & 0 \\ -d'_2 & q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases} \qquad \begin{pmatrix} a_3 & b_3 & c_3 \\ d_3 & e_3 & f_3 \\ g_3 & h_3 & i_3 \end{pmatrix} = MH_1H_2H_3
$$

$$
H_4 = \begin{cases} I_3 & \text{if } a_3 > 0 \text{ and } d_3 > 0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } a_3 > 0 \text{ and } d_3 < 0 \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } a_3 < 0 \text{ and } d_3 > 0 \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}
$$

with $g'_i = \frac{g_i}{\gcd(g_i,h_i)}$ *and* $h'_i = \frac{h_i}{\gcd(g_i,h_i)}$, u_i *and* v_i *such that* $u_i g_i + v_i h_i = \gcd(g_i,h_i)$. $d'_2 = \frac{d_2}{\gcd(d_2,e_2)}$, $e'_2 = \frac{e_2}{\gcd(d_2,e_2)}$ *and* p_2, q_2 *such that* $p_2 d_2 + q_2 e_2 = \gcd(d_2,e_2)$.

Proof. We do not give here all the details of the Hermite Normal Form in 3-D (very similar to the 2-D case). Using the notations of the lemma, we can check that: each matrix is such that each matrix H_i is such that $|det(H_i)| = 1$. Furthermore, we can check that each matrix H_i ($i \in \{1, \dots 3\}$) eliminates one coefficient leading to an upper triangular matrix.

Finally, the last matrix H_4 is here to make the diagonal coefficient positive.□

A.16 Theorem 11

Proof. Since $a > 0$ (Hermite Normal Form), we have $\alpha_h > 0$. First, $MU = TY = \begin{pmatrix} a\omega'_h \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a'_h\omega \\ 0 \\ 0 \end{pmatrix} = \omega \begin{pmatrix} \alpha_h \\ 0 \\ 0 \end{pmatrix}$ Let $X \in \mathbb{Z}^3$,

$$
\begin{aligned}
&M(X-U)+V \\
&\qquad = MX + V - MU \\
&\qquad = \omega \left\lceil \frac{MX+V}{\omega} \right\rceil + \left\{ \frac{MX+V}{\omega} \right\} - \omega \begin{pmatrix} \alpha_h \\ 0 \\ 0 \end{pmatrix} \\
&\qquad = \omega \left(\left\lceil \frac{MX+V}{\omega} \right\rceil - \begin{pmatrix} \alpha_h \\ 0 \\ 0 \end{pmatrix} \right) + \left\{ \frac{MX+V}{\omega} \right\}
\end{aligned}
$$

Hence, $\left\lceil \frac{M(X-U)+V}{\omega} \right\rceil = \left\lceil \frac{MX+V}{\omega} \right\rceil - \begin{pmatrix} \alpha_h \\ 0 \\ 0 \end{pmatrix}$ and $\left\{ \frac{M(X-U)+V}{\omega} \right\} = \left\{ \frac{MX+V}{\omega} \right\}$. Let $(i,j,k) \in \mathbb{Z}^3$, if $X \in P_{i+\alpha_h,j,k}$, we have $X - U \in P_{i,j,k}$ and therefore $P_{i+\alpha_h,j,k} \equiv P_{i,j,k}$. This also gives the vector of translation : $X - (X-U) = U$

A.17 Theorem 12

Proof. Let us first prove that α_h divides α_0. Using Def. 8, we have $\forall (i,j,k) \in \mathbb{Z}^3, P_{i+\alpha_0,j,k} = T_U P_{i,j,k}$ Given $(i,j,k) \in \mathbb{Z}^3$, $X \in P_{i+\alpha_0,j,k}$ and $X' = X - U$, we have $X' \in P_{i,j,k}$ and $\left\{ \frac{MX+V}{\omega} \right\} = \left\{ \frac{MX'+V}{\omega} \right\}$. If we denote $H^{-1}U = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we obtain

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= MU = M(X - X') \\
&= (MX + V) - (MX' + V) \\
&= \omega \begin{pmatrix} i + \alpha_0 \\ j \\ k \end{pmatrix} - \omega \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \omega \begin{pmatrix} \alpha_0 \\ 0 \\ 0 \end{pmatrix} ,
\end{aligned}
$$

which implies

$$
\begin{cases} ax + by + cz = \omega\alpha_0 \\ ey + fz = 0 \\ iz = 0 \end{cases} . \tag{13}
$$

Thanks to the Hermite Normal Form, $i \neq 0$, hence we have $z = 0$. Then,

$$
(13) \Rightarrow \begin{cases} ax + by = \omega\alpha_0 \\ ey = 0 \end{cases}
$$

but $e \neq 0$, hence $y = 0$. Then,

$$
ax = \omega\alpha_0 \Rightarrow a'_h x = \omega'_h \alpha_0 \Rightarrow a'_h | \omega'_h \alpha_0, \text{ or } \gcd(a'_h, \omega'_h) = 1 .
$$

Thanks to the Gauss Theorem, $a'_h | \alpha_0$ and thus $\alpha_h | \alpha_0$. Finally, by definition of α_0, $\alpha_h = \alpha_0$.□

A.18 Theorem 13

Proof. First, since $d > 0$, we have $\alpha_v > 0$.

$$
\begin{aligned}
MU = TY &= \begin{pmatrix} -a\phi u_1 + b\omega'_v \alpha'_v \\ d\omega'_v \alpha'_v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -a'_v b\omega'_v u_1 + b\omega'_v \alpha'_v \\ d'_v \omega \alpha'_v \\ 0 \end{pmatrix} = \begin{pmatrix} b\omega'_v(-a'_v u_1 + \alpha'_v) \\ \omega\alpha_v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} b\omega'_v \omega''_v v_1 \\ \omega\alpha_v \\ 0 \end{pmatrix} = \begin{pmatrix} \phi\omega v_1 \\ \omega\alpha_v \\ 0 \end{pmatrix} = \omega \begin{pmatrix} -\beta_0 \\ \alpha_v \\ 0 \end{pmatrix}
\end{aligned}
$$

Hence, $\left[\frac{M(X-U)+V}{\omega}\right] = \left[\frac{MX+V}{\omega}\right] - \begin{pmatrix} -\beta_0 \\ \alpha_v \\ 0 \end{pmatrix}$ and $\left\{\frac{M(X-U)+V}{\omega}\right\} = \left\{\frac{MX+V}{\omega}\right\}$.
Let $(i,j,k) \in \mathbb{Z}^3$, if $X \in P_{i,j+\alpha_v,k}$, we have $X - U \in P_{i+\beta_0,j,k}$ and therefore $P_{i,j+\alpha_v,k} \equiv P_{i+\beta_0,j,k}$. This also gives the vector of translation : $X-(X-U) = U$□

A.19 Theorem 14

Proof. Let us first prove that α divides α_1 . Using Def. 8, we have

$$\forall(i,j,k) \in \mathbb{Z}^3, P_{i,j+\alpha_1,k} = T_{U_1} P_{i+\beta_0^1,j,k}$$

Let $(i,j,k) \in \mathbb{Z}^3$, $X \in P_{i,j+\alpha_1,k}$ and $X' = X - U_1$, then $X' \in P_{i+\beta_0^1,j,k}$, and $\left\{\frac{MX+V}{\omega}\right\} = \left\{\frac{MX'+V}{\omega}\right\}$. Let us denote

$$H^{-1}U_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence,

$$\begin{aligned} T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= MU_1 = M(X - X') = (MX + V) - (MX' + V) \\ &= \omega \begin{pmatrix} i \\ j + \alpha_1 \\ k \end{pmatrix} - \omega \begin{pmatrix} i + \beta_0^1 \\ j \\ k \end{pmatrix} = \omega \begin{pmatrix} -\beta_0^1 \\ \alpha_1 \\ 0 \end{pmatrix}, \end{aligned}$$

which implies

$$\begin{cases} ax + by + cz = -\omega\beta_0^1 \\ dy + ez = \omega\alpha_1 \\ fz = 0 \end{cases} \tag{14}$$

$$\begin{aligned} (14) &\Rightarrow z = 0, \\ &\Rightarrow d'_v y = \omega'_v \alpha_1 \\ &\Rightarrow d'_v | \omega'_v \alpha_1, \text{ or } \gcd(d'_v, \omega'_v) = 1 \\ &\Rightarrow d'_v | \alpha_1 \end{aligned}$$

Let $\alpha_1 = d'\alpha'_1$, then we have $y = \omega'_v \alpha'_1$, we have

$$\begin{aligned} & ax + b\omega'_v \alpha'_1 = -\omega\beta_0^1 \\ & \Rightarrow a'_v x + \phi\alpha'_1 = -\omega''_v \beta_0^1 \\ & \Rightarrow \alpha' = \gcd(a'_v, \omega''_v) | \phi\alpha'_1, \text{ or } \gcd(a'_v, \phi, \omega''_v) = 1 \\ & \Rightarrow \alpha'_v | \alpha'_1 \\ & \Rightarrow \alpha_v | \alpha_1 \end{aligned}$$

Finally, by definition of α_1, $\alpha_v = \alpha_1$.$\square$

A.20 Theorem 15

Proof. Since $f > 0$, then $\alpha_d > 0$.

$$
\begin{aligned}
MU &= TY \\
&= \begin{pmatrix} -a\psi' u_2 - b\phi u_1 \alpha''_d + bk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} + c\omega'_d\alpha'_d \\ -d\phi u_1\alpha''_d + dk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} + e\omega'_d\alpha'_d \\ f\omega'_d\alpha'_d \end{pmatrix} \\
&= \begin{pmatrix} -a'_d\psi u_2 - b\phi u_1 \alpha''_d + bk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} + c\omega'_d\alpha'_d \\ -d'_d e\omega'_d u_1\alpha''_d + dk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} + e\omega'_d\alpha'_d \\ f'_d\omega\alpha'_d \end{pmatrix} \\
&= \begin{pmatrix} -\psi(\alpha''_d - \chi v_2 - \omega'''_d w_2) - b\phi u_1\alpha''_d + bk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} \\ \hfill + c\omega'_d\alpha'_d \\ -e\omega'_d(\gcd(d'_d,\omega''_d) - \omega''_d v_1)\alpha''_d + dk\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} \\ \hfill + e\omega'_d\alpha'_d \\ \omega\alpha_d \end{pmatrix} \\
&= \begin{pmatrix} -\psi\alpha''_d + \psi'\frac{\omega''_d b}{\gcd(d'_d,\omega''_d)} v_2 + \psi'\omega w_2 - b\phi u_1\alpha''_d \\ \hfill -b\psi' v_2\frac{\omega''_d}{\gcd(d'_d,\omega''_d)} + c\omega'_d\alpha'_d \\ -e\omega'_d \gcd(d'_d,\omega''_d)\alpha''_d + \phi\omega v_1\alpha''_d + d'_d k\frac{\omega}{\gcd(d'_d,\omega''_d)} \\ \hfill + e\omega'_d\alpha''_d \gcd(d'_d,\omega''_d) \\ \omega\alpha_d \end{pmatrix} \\
&= \begin{pmatrix} -(c\omega'_d \gcd(d'_d,\omega''_d) - b\phi u_1)\alpha''_d + \psi'\omega w_2 - b\phi u_1\alpha''_d \\ \hfill + c\omega'_d\alpha''_d \gcd(d'_d,\omega''_d) \\ -\omega\beta_1 \\ \omega\alpha_d \end{pmatrix} = \omega\begin{pmatrix} -\beta_0 \\ -\beta_1 \\ \alpha_d \end{pmatrix}
\end{aligned}
$$

Hence, $\left[\frac{M(X-U)+V}{\omega}\right] = \left[\frac{MX+V}{\omega}\right] - \begin{pmatrix} -\beta_0 \\ -\beta_1 \\ \alpha_d \end{pmatrix}$ and $\left\{\frac{M(X-U)+V}{\omega}\right\} = \left\{\frac{MX+V}{\omega}\right\}$.
Let $(i,j,k) \in \mathbb{Z}^3$, if $X \in P_{i,j,k+\alpha_d}$, we have $X - U \in P_{i+\beta_0,j+\beta_1,k}$ and therefore $P_{i,j,k+\alpha_d} \equiv P_{i+\beta_0,j+\beta_1,k}$. This also gives the vector of translation : $X-(X-U) = U$ $\square$

A.21 Theorem 16

Proof. Let us first prove that α_d divides α_2 . Using Def. 8, we have

$$\forall (i,j,k) \in \mathbb{Z}^3, P_{i,j,k+\alpha_2} = T_{U_2} P_{i+\beta_0^2,j+\beta_1^2,k}$$

Let $(i,j,k) \in \mathbb{Z}^3$, $X \in P_{i,j,k+\alpha_2}$ and $X' = X - U_2$, then $X' \in P_{i+\beta_0^2, j+\beta_1^2, k}$ and $\left\{\frac{MX+V}{\omega}\right\} = \left\{\frac{MX'+V}{\omega}\right\}$. Let us denote

$$H^{-1}U_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

Hence,

$$\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= MU_2 = M(X - X') \\
&= (MX+V) - (MX'+V) = \omega \begin{pmatrix} i \\ j \\ k+\alpha_2 \end{pmatrix} - \omega \begin{pmatrix} i+\beta_0^2 \\ j+\beta_1^2 \\ k \end{pmatrix} \\
&= \omega \begin{pmatrix} -\beta_0^2 \\ -\beta_1^2 \\ \alpha_2 \end{pmatrix} ,
\end{aligned}$$

which implies

$$\begin{cases} ax + by + cz = -\omega\beta_0^2 \\ dy + ez = -\omega\beta_1^2 \\ fz = \omega\alpha_2 \end{cases} . \tag{15}$$

$$\begin{aligned}
Eq.(15) &\Rightarrow f'_d z = \omega'_d \alpha_2 \\
&\Rightarrow f'_d | \omega'_d \alpha_2, \text{ or } \gcd(f'_d, \omega'_d) = 1 \\
&\Rightarrow f'_d | \alpha_2
\end{aligned}$$

with $\alpha_2 = f'\alpha'_2$, we have $z = \omega'_d \alpha'_2$, and

$$\begin{aligned}
&\Rightarrow dy + e\omega'_d \alpha'_2 = -\omega\beta_1^2 \\
&\Rightarrow d'_d y + \phi\alpha'_2 = -\omega''_d \beta_1^2 \\
&\Rightarrow \gcd(d'_d, \omega''_d) | \phi\alpha'_2, \text{ or } \gcd(d'_d, \phi, \omega''_d) = 1 \\
&\Rightarrow \gcd(d'_d, \omega''_d) | \alpha'_2
\end{aligned}$$

With $\alpha'_2 = \gcd(d'_d, \omega''_d)\alpha''_2$, we have

$$\begin{aligned}
&\Rightarrow d'_d y + \omega''_d \beta_1^2 = -\phi \gcd(d'_d, \omega''_d)\alpha''_2, \\
&\qquad\qquad \text{or } d'_d(-\phi\alpha''_2 u_1) + \omega''_d(-\phi\alpha''_2 v_1) = -\phi \gcd(d'_d, \omega''_d)\alpha''_2, \\
&\Rightarrow \frac{d'_d}{\gcd(d'_d, \omega''_d)}(y + \phi\alpha''_2 u_1) = \frac{\omega''_d}{\gcd(d'_d, \omega''_d)}(\beta_1^2 + \phi \gcd(d'_d, \omega''_d)\alpha''_2) \\
&\Rightarrow \frac{\omega''_d}{\gcd(d'_d, \omega''_d)} \Big| \frac{d'_d}{\gcd(d'_d, \omega''_d)}(y + \phi\alpha''_2 u_1), \\
&\qquad\qquad \text{or } \gcd\left(\frac{d'_d}{\gcd(d'_d, \omega''_d)}, \frac{\omega''_d}{\gcd(d'_d, \omega''_d)}\right) = 1,
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \frac{\omega''_d}{\gcd(d'_d,\omega''_d)}|(y+\phi\alpha''_2 u_1)\\
&\Rightarrow \exists k'\in\mathbb{Z}/y=-\phi\alpha''_2u_1+k'\frac{\omega''_d}{\gcd(d'_d,\omega''_d)},\\
&\Rightarrow ax-b\phi\alpha''_2u_1+bk'\frac{\omega''_d}{\gcd(d'_d,\omega''_d)}+c\omega'_d\alpha'_2=-\omega\beta_0^2,\\
&\Rightarrow ax-b\phi u_1\alpha''_2+bk'\frac{\omega''_d}{\gcd(d'_d,\omega''_d)}+c\omega'_d\gcd(d'_d,\omega''_d)\alpha''_2=-\omega\beta_0,\\
&\Rightarrow ax+\psi\alpha''_2+k'\frac{b\omega''_d}{\gcd(d'_d,\omega''_d)}+\omega\beta_0^2=0,\\
&\Rightarrow a'_dx+k'\chi+\omega'''_d\beta_0^2=-\psi'\alpha''_2,\\
&\Rightarrow \alpha''_d=\gcd(a'_d,\chi,\omega'''_d)|\psi'\alpha''_2,\ \text{ or }\ \gcd(a'_d,\psi,\chi,\omega'''_d)=1,\\
&\Rightarrow \alpha''_d|\alpha''_2,\\
&\Rightarrow \alpha_d|\alpha_2
\end{aligned}
$$

Finally, by definition of α_2, $\alpha_d=\alpha_2$.□

A.22 Super-paving construction details

Using Theorems 5 and 6, we have

$$\theta_0=\frac{\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\alpha_0},\theta_1=\frac{\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\alpha_1},\theta_1=\frac{\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\alpha_2},$$

and

$$\mathcal{P}=\left\{H\begin{pmatrix}x\\y\\z\end{pmatrix}/A'_2\le z<B'_2, A'_1(z)\le y<B'_1(z)\text{ and }A'_0(y,z)\le x<B'_0(y,z)\right\},$$

with

$$A'_2=-\left[\frac{\theta_2l_0}{\theta_2f}\right],B'_2=-\left[\frac{-\omega\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)+\theta_2l_0}{\theta_2f}\right]$$

$$A'_1(z)=-\left[\frac{\theta_1k_0+\theta_1ez}{\theta_1d}\right],B'_1(z)=-\left[\frac{-\omega\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)+\theta_1k_0+\theta_1ez}{\theta_1d}\right]$$

$$A'_0(y,z)=-\left[\frac{\theta_0j_0+\theta_0by+\theta_0cz}{\theta_0a}\right],B'_0(y,z)=-\left[\frac{-\omega\operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)+\theta_0j_0+\theta_0by+\theta_0cz}{\theta_0a}\right]$$

Hence, $A'_2=-\left[\frac{l_0}{f}\right]$, $A'_1(z)=-\left[\frac{k_0+ez}{d}\right]$, $A'_0(y,z)=-\left[\frac{j_0+by+cz}{a}\right]$, $B'_2=A'_2+\frac{\omega\alpha_2}{f}$, $B'_1(z)=A'_1(z)+\frac{\omega\alpha_1}{d}$, and $B'_0(y,z)=A'_0(y,z)+\frac{\omega\alpha_0}{a}$.

Furthermore, using minimal periodicity notations, we have

$$\frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\theta_2 f} = \frac{\omega\alpha_2}{f} = \frac{\omega'_h\alpha_2}{f'_h} = \omega'_h\alpha'_2 \in \mathbb{Z}$$
$$\frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\theta_1 d} = \frac{\omega\alpha_1}{d} = \frac{\omega'_v\alpha_1}{d'_v} = \omega'_v\alpha'_1 \in \mathbb{Z}$$
$$\frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\theta_0 a} = \frac{\omega\alpha_0}{a} = \frac{\omega'_d\alpha_0}{a'_d} = \omega'_d \in \mathbb{Z}\,.$$

Thus, we obtain

$$B'_2 = \frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\theta_2 f} - \left[\frac{l_0}{f}\right] = A'_2 + \frac{\omega\alpha_2}{f},$$
$$B'_1(z) = \frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{\theta_1 d} - \left[\frac{k_0+ez}{d}\right] = A'_1(z) + \frac{\omega\alpha_1}{d}$$
$$B'_0(y,z) = \frac{\omega \operatorname{lcm}(\alpha_0,\alpha_1,\alpha_2)}{a} - \left[\frac{j_0+by+cz}{a}\right] = A'_0(y,z) + \frac{\omega\alpha_0}{a}$$

A.23 Proposition 3

Proof. The number of pavings of the initial period is equal to the number of points in the super-paving. Hence,

$$\begin{aligned}
\frac{\omega\alpha_0}{a}\frac{\omega\alpha_1}{d}\frac{\omega\alpha_2}{f} &= \frac{\omega}{\gcd(a,\omega)}\frac{\omega\gcd(a'_v,\omega''_v)}{\gcd(d,\omega)}\frac{\omega\gcd(a'_d,\chi,\omega'''_d)\gcd(d'_d,\omega''_d)}{\gcd(f,\omega)}\\
&= \frac{\omega}{\gcd(a,\omega)}\frac{\omega\gcd(a,\omega)}{\gcd(d,\omega)\gcd(a,b\omega'_v,\omega)}\frac{\omega\gcd(a'_d,\chi,\omega'''_d)\gcd(d,\omega)}{\gcd(f,\omega)\gcd(d,e\omega'_d,\omega)}\\
&= \omega\frac{\omega}{\gcd(a,b\omega'_v,\omega)}\frac{\omega\gcd(a'_d,\chi,\omega'''_d)}{\gcd(f,\omega)\gcd(d,e\omega'_d,\omega)}\\
&= \omega'_d\omega''_d\frac{\omega\gcd(a'_d,\chi,\omega'''_d)}{\gcd(a,b\omega'_v,\omega)}\\
&= \omega'_d\omega''_d\frac{\omega\gcd(a,\frac{b\omega''_d}{\gcd(d'_d,\omega''_d)},\omega)}{\gcd(a,b\frac{\omega}{\gcd(d,\omega)},\omega)\gcd(a,\psi,\omega,\frac{b\omega''_d}{\gcd(d'_d,\omega''_d)})}\\
&= \omega'_d\omega''_d\frac{\omega\gcd(a,\frac{b\omega}{\gcd(d,\omega)},\omega)}{\gcd(a,b\frac{\omega}{\gcd(d,\omega)},\omega)\gcd(a,\psi,\omega,\frac{b\omega''_d}{\gcd(d'_d,\omega''_d)})}\\
&= \omega'_d\omega''_d\frac{\omega}{\gcd(a,\psi,\omega,\frac{b\omega''_d}{\gcd(d'_d,\omega''_d)})}\\
&= \omega'_d\omega''_d\omega'''_d
\end{aligned}$$

□

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