Discrete Circularity Measure

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Abstract. A new circularity measure for discrete objects is introduced and an algorithm to compute this measure is presented. This measure equals 0 for all discrete circles and is strictly positive for all discrete objects which are not discrete circles.

1 Introduction

The study of geometric properties of discrete objects, *i.e.* connected sets of discrete points, for shape recognition or shape description, is an important topic of Discrete Geometry. A classical way to describe discrete objects shapes is to compute global estimators like perimeter, area, eccentricity, compactness, etc. See [Lon98] or [ZL04] for a review of shape representation and description techniques.

In this paper, we focus on circularity. A classical circularity measure in the Euclidean plane is $P^2/A - 4\pi$ where A is the area and P the perimeter. The discrete equivalent of this circularity measure was introduced in [Har74], but coarse estimation of perimeter makes the measure unsatisfactory: discrete circles may have neither the same circularity value nor the smallest circularity value. Since [Har74], lots of progress have been done on length estimation. In [DT03], it has been shown that length estimations computed with local definition of length are never convergent. In [CK04], it has been shown that perimeter estimation based on DSS recognition (see [DRR95]) is convergent and behave very well with respect to other perimeter estimation methods. However, even with these improvements, one problem remains: discrete circles have not the same circularity value.

To our knowledge, only one paper dealt with this problem, more than twenty years ago. In [KA84], a discrete disk recognition algorithm in $\mathcal{O}(n^2)$ is presented in the first part, and a discrete compactness evaluation algorithm for discrete convex objects in $\mathcal{O}(n^3\sqrt{n})$ is presented in the second part (where n is the number of pixels of the discrete curve). The new discrete compactness measure is the ratio between area A and area A' of the smallest enclosing discrete disk (where "the smallest" is expressed in area unit, *i.e.* in number of pixels). As a smallest enclosing discrete disk may not be unique and as the smallest enclosing

* Supported by a grant from the french DGA (Jacques Blanc-Talon).

euclidean disk may not be a smallest enclosing discrete disk, areas of many discrete disks have to be compared. This is why the computational cost is rather high. This first attempt show us that the problem is not trivial because none point such as the center of the smallest enclosing circle, the center of the biggest inscribed circle or the centroid, are always a circle center, the discretization of which is the discrete object to be described.

Since [KA84], lots of progress have been done in discrete disk or circle recognition, but not in discrete circularity computation, whereas both problems are linked. Several authors based their methods on the classical separating arc problem in Computational Geometry. Fisk [Fis86] presents an algorithm to detect if a set of grid points in a $N \times N$ image is a discrete disk in $\mathcal{O}(N^2)$. In addition, this algorithm constructs the complete solution domain, contrary to Kim's one [KA84]. Based on a similar approach, Kovalevsky [Kov90] proposes a discrete circle recognition, the computational cost of which is estimated by Coeurjolly *et al.* [CGRT04] in $\mathcal{O}(n^2.log(n))$. Coeurjolly *et al.* [CGRT04] solve the classical separating arc problem using classical tools and improve the computational cost due to an arithmetical approach. Their algorithm, which may be incremental, constructs the complete solution domain in $\mathcal{O}(n^{4/3}.log(n))$. However, it is possible to reach a linear time computional cost, giving up the incremental construction of the complete solution domain. Damaschke [Dam95] proves that the separating arc problem in one quadrant is equivalent to solve a set of $2n$ inequalities in dimension 3. Thus, the Megiddo's [Meg84] algorithm can be used to decide if a discrete curve is a discrete circle in $\mathcal{O}(n)$.

The objective of the paper is to propose a consistent discrete circularity measure (*i.e.* the measure equals 0 for all discrete circles). To do this, our approach is to transform the boolean output of an existing discrete circle recognition algorithm into a real output.

The paper is organised as follows. In section 2, we recall the algorithm presented in [CGRT04] to solve the separating arc problem. In section 3, we introduce a new discrete circularity measure, prove that it gives 0 for all discrete circles, and presente the algorithm which computes it. Experiments are given in section 4 with synthetical images. The paper ends with some conclusions and future works in section 5.

2 The separating arc problem

In this section, we recall the algorithm presented in [CGRT04] to solve the separating arc problem, and we extend it in order to compute a new discrete circularity measure. We opted for the notation used in [CGRT04].

Let $\mathcal{S}$ ant $\mathcal{T}$ be two finite sets of points of $\mathbb{Z}^2$. $\mathcal{S}$ is called *circularly separable* from $\mathcal{T}$ if there exists an Euclidean disk $\mathcal{C}(\omega, R)$ centered at ω and with radius R, such that: $\forall s \in \mathcal{S}$, $\forall t \in \mathcal{T}$, $s \in \mathcal{C}$ *and* $t \notin \mathcal{C}$.

If such a circle exists for given $\mathcal{S}$ and $\mathcal{T}$, then $\omega \in acd(\mathcal{S}, \mathcal{T})$, where $acd(\mathcal{S}, \mathcal{T})$, nammed the *arc center domain*, is such that:

$$acd(\mathcal{S},\mathcal{T}) = \{\omega | \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, s \in \mathcal{C} \text{ and } t \notin \mathcal{C}\}$$

$$\Leftrightarrow acd(\mathcal{S},\mathcal{T}) = \bigcap_{\forall s \in \mathcal{S}, \forall t \in \mathcal{T}} \mathcal{H}(s,t)$$

where $\mathcal{H}(s,t)$ is the half-plane bounded by bisector of $[st]$ and containing s.

$$\Leftrightarrow acd(\mathcal{S},\mathcal{T}) = \bigcap_{\forall s \in \mathcal{S}, \forall t \in \mathcal{T}} \{P | \vec{st}.\vec{MP} < 0\}$$

where M is the midpoint of $[st]$.

$$\Leftrightarrow acd(s,\mathcal{T}) = \bigcap_{\forall s \in \int, \forall t \in \mathcal{T}} \{(x_p,y_p) | 2(x_t - x_s)x_p + 2(y_t - y_s)y_p + {x_s}^2 + {y_s}^2 - {x_t}^2 - {y_t}^2 < 0\}$$

$$\Leftrightarrow acd(s,\mathcal{T}) = \bigcap_{\forall s \in \int, \forall t \in \mathcal{T}} \{(x,y) | a.x + b.y + c < 0\}$$

where $P(x_p = x, y_p = y),\ a = 2(x_t - x_s),\ b = 2(y_t - y_s),\ c = {x_s}^2 + {y_s}^2 - {x_t}^2 - {y_t}^2$.

Given a discrete curve, the authors of [CGRT04] show how to linearly compute a small set of pixels $\mathcal{S}$ and $\mathcal{T}$ such that $\mathcal{S}$ is circularly separable from $\mathcal{T}$ if and only if the discrete curve to process is the OBQ digitization of a circle (Proposition 6 in [CGRT04]). $\mathcal{S}$ is the convex hull of the discrete curve, whereas points $t \in \mathcal{T}$ are the closest points to the middle of $[s_i s_j]$, an edge of $\mathcal{S}$. Figure 2(a) and (b) give an illustration of digital curves together with the sets $\mathcal{T}$ computed after [CGRT04].

Let us suppose that $b \neq 0$. Each half-plane $\mathcal{H}(s,t)$ is bounded by a straight line having equation $y = -\frac{a}{b}.x - \frac{c}{b}$. This straight line is the bisector of s and t. Each half-plane $\mathcal{H}(s,t)$, containing s but not t, is :

- the set of points located *below* its boundary when $b > 0$, such that:

$$\{(x,y) | y < -\frac{a}{b}.x - \frac{c}{b}\}$$

- the set of points located *above* its boundary when $b < 0$, such that:

$$\{(x,y) | y > -\frac{a}{b}.x - \frac{c}{b}\}$$

As in [CGRT04], we use a dual space to represent these sets of inequalities. However, where the authors use this transformation only to improve the computation of $acd(s,t)$, we propose to go further in its use to propose a new measure of circularity.

Figure 1 recalls the definition of the classical point-line duality. In our case, we do not deal with straight lines but with inequalities that are represented by oriented half-planes.

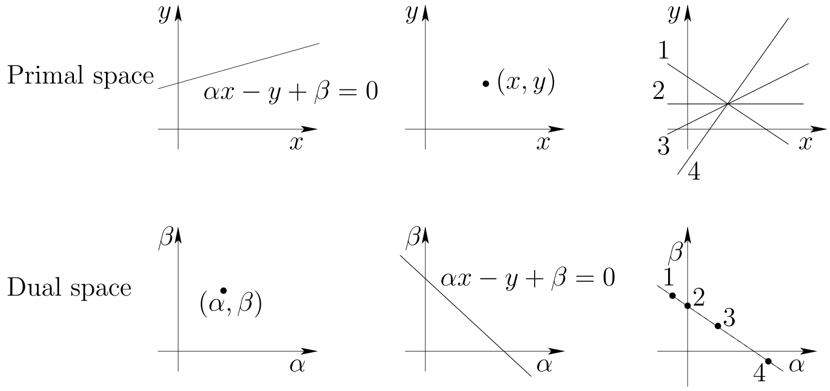


Fig. 1. Point-line duality : a line of the primal space is represented by a point in the dual space and conversely. Three lines intersecting in the primal space map to three colinear points in the dual space.

We define the dual of a half-plane $\mathcal{H} = \{(x, y) | y < -\frac{a}{b}.x - \frac{c}{b}\}$ as the labelled point $\mathcal{H}^{l^*}(-\frac{a}{b}, \frac{c}{b})$, where l = '+' when $b > 0$ and l = '-' when $b < 0$. Similarly, we denote $N^* = \{(-\frac{a}{b}, \frac{c}{b}) | \frac{c}{b} < -\frac{a}{b}.x - y\}$ the half-plane dual of labelled point $N^l(-\frac{a}{b}, \frac{c}{b})$. See Figure 2 for an illustration of this transformation. In lemma 1, we suppose that $b > 0$, and omit the label l in the notations. The case where $b < 0$ is symmetric. Proposition 1 of [CGRT04] is stated in a different way in the following lemma :

Lemma 1. *$N \in \mathcal{H}$ if and only if $\mathcal{H}^* \in N^*$.*

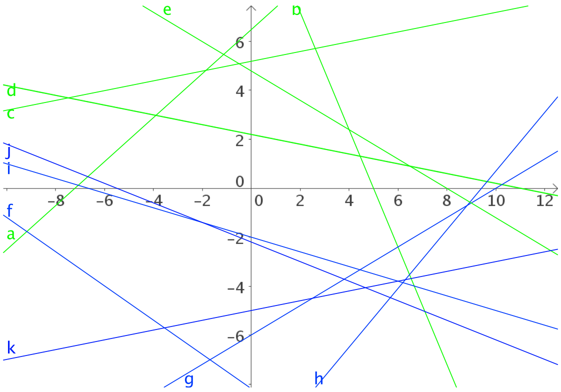
Proof.

$$\begin{gathered} N \in \mathcal{H} \\ \Leftrightarrow \bigcap \{N(x, y) | y < -\frac{a}{b}.x - \frac{c}{b}\} \\ \Leftrightarrow \bigcap \{\mathcal{H}(-\frac{a}{b}, \frac{c}{b}) | \frac{c}{b} < -\frac{a}{b}.x - y\} \\ \Leftrightarrow \mathcal{H}^* \in N^* \end{gathered}$$

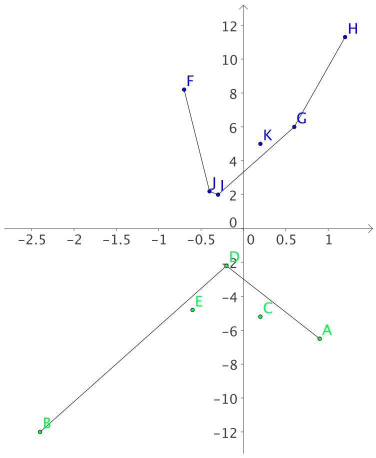
□

As stated above, two kinds of inequalities may be defined according to the sign of b. Let us denote $\mathcal{K}^+$ (resp. $\mathcal{K}^-$) the set of half-planes $\mathcal{H}$ where $b > 0$ (resp. $b < 0$). Let us also denote $\mathcal{L}^+ = \bigcap \mathcal{H}, \forall \mathcal{H} \in \mathcal{K}^+$ (resp. $\mathcal{L}^- = \bigcap \mathcal{H}, \forall \mathcal{H} \in \mathcal{K}^-$). The image in the dual space of the set $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is a set of points denoted by $\mathcal{L}^{+^*}$ (resp. $\mathcal{L}^{-^*}$). $\mathcal{L}^{+^*}$ (resp. $\mathcal{L}^{-^*}$) is the upper convex hull (rep. lower convex hull) of the points of $\mathcal{K}^{+^*}$ (resp. $\mathcal{K}^{-^*}$). Figure 2 shows the link between intersection of half-planes in the primal space and convex hull of points in the dual space.

Lemma 2. *$\mathcal{L}^+ \bigcap \mathcal{L}^- \neq \emptyset$ if and only if $\mathcal{L}^{+^*} \bigcap \mathcal{L}^{-^*} = \emptyset$. Moreover, $\mathcal{L}^{+^*}$ is* below *$\mathcal{L}^{-^*}$.*



(a)



(b)

Fig. 2. Half-plane/labelled point duality : (a) a set of oriented half-planes. Half-planes labelled from a to e (resp. f to k) are such that $b > 0$ (resp. $b < 0$) ; (b) dual points of the half-planes, labelled with corresponding upper-case letters.

Proof. Suppose that $\mathcal{L}^+ \bigcap \mathcal{L}^- \neq \emptyset$. Then there exists a point N such that $N \in \mathcal{L}^+ \bigcap \mathcal{L}^-$. Applying lemma 1, it is clear that $N^+ \in \mathcal{L}^+ \Leftrightarrow \mathcal{L}^{+^*} \in N^{+^*}$ *i.e.* the whole set of points $\mathcal{L}^{+^*}$ is located *below* (since $b > 0$) the boundary straight line of half-plane N^{+^*}. It is also clear that $N^- \in \mathcal{L}^- \Leftrightarrow \mathcal{L}^{-^*} \in N^{-^*}$ *i.e.* the whole set of points $\mathcal{L}^{-^*}$ is located *above* (since $b < 0$) the boundary straight line of half-plane N^{-^*}. Consequently, there exists a half-plane N^{+^*} (or N^{-^*}) that contains $\mathcal{L}^{+^*}$ but not $\mathcal{L}^{-^*}$ (or $\mathcal{L}^{-^*}$ but not $\mathcal{L}^{+^*}$), *i.e* $\mathcal{L}^{+^*} \bigcap \mathcal{L}^{-^*} = \emptyset$. The reverse implication is proved in the same way. □

Lemma 2 shows how to detect if $\mathcal{S}$ is circularly separable from $\mathcal{T}$:

- compute $\mathcal{L}^{+^*}$ and $\mathcal{L}^{-^*}$;
- check if the two polygonal lines intersect or not. If not, then $\mathcal{S}$ is circularly separable from $\mathcal{T}$.

We can derive the following property from Lemma 2 :

Corollary 1. *A discrete curve is a discrete circle if and only if $\mathcal{L}^{+^*}$ and $\mathcal{L}^{-^*}$ do not intersect each other. Otherwise, $\mathcal{L}^{+^*} \cap \mathcal{L}^{-^*}$ is a non-empty convex polygon.*

This property is used in the following to define a new circularity measure.

3 A new discrete circularity measure

3.1 Definitions

Let us recall that $\mathcal{H}(s,t)$ ($\mathcal{H}$ for short) is the half-plane bounded by bisector of s and t and containing s. $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is the intersection of half-planes $\mathcal{H}$ where

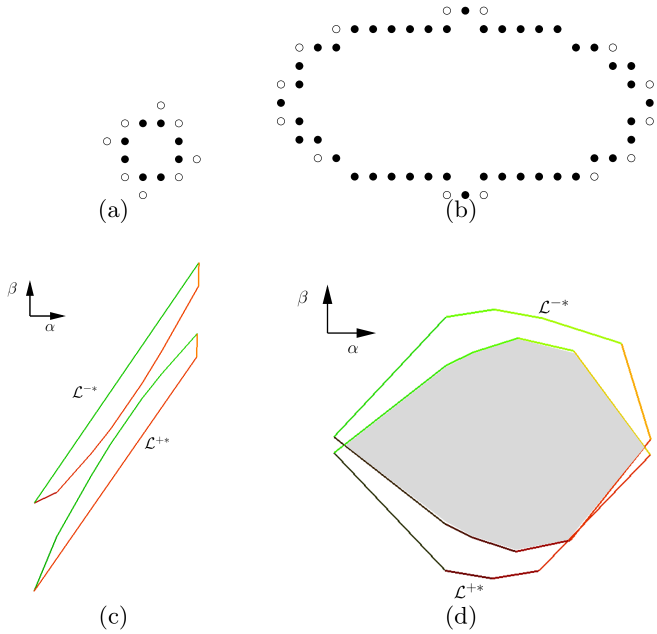


Fig. 3. $\mathcal{S}$ is the convex hull of the set of black points (the discrete curve). $\mathcal{T}$ is the set of white points. Sets $\mathcal{L}^{+*}$ and $\mathcal{L}^{-*}$ in the dual space are depicted respectively in c) and d) for discrete curves drawn respectively in a) and b).

$b > 0$ (resp. $b < 0$) and $\mathcal{L}^{+*}$ (resp. $\mathcal{L}^{-*}$) the convex hull of points $\mathcal{H}^*$, where x^* is the dual of x.

Definition 1. *Given a discrete curve, we call* circularity *the vertical width of* $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*}$. *If* $\mathcal{L}^{+*}$ *and* $\mathcal{L}^{-*}$ *do not intersect each other, the circularity is set to zero.*

The vertical width of a polygon is the minimum distance between two horizontal parallel lines encompassing the polygon.

Property 1. The circularity is ranging from 0 to $+\infty$. Moreover, the cirularity is 0 if and only if the digital curve $\mathcal{S}$ is the OBQ digitization of a circle.

This property is straightforward from Corollary 1 : $\mathcal{S}$ is a digital circle if and only if $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*} = \emptyset$, and thus the cirularity is 0. Otherwise $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*} \neq \emptyset$ and the vertical width of the intersection polygon is a strictly positive value.

The idea under this definition is to associate the distance from a discrete curve to a discrete circle with a measure of the polygon $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*}$. Among all the measures one could imagine, the vertical distance has the nice following property:

Property 2. Let l and v respectively be a line and a vertex, and denote l^* and v^* their respective images in the dual space. Then $d(l^*, v^*) = d(l, v)$, where d denotes the vertical distance.

This property implies that the vertical width of $\mathcal{L}^{+^*} \bigcap \mathcal{L}^{-^*}$ can be geometrically interpreted in the primal space where the bisectors are defined. Figure 4 illustrates this interpretation: if the vertical width is attained between the point l^* and the straight line v^* (see (a)), then if v^* is moved vertically by a distance $d' > d(l^*, v^*)$ (see (b)), $\mathcal{L}^{+^*} \bigcap \mathcal{L}^{-^*}$ becomes empty. Similarly, in the primal space, the distance between the upper convex hull $\mathcal{L}^+$ and the lower convex hull $\mathcal{L}^-$ is also the vertical width $d(l^*, v^*)$ from Property 2, and the sets of constraints becomes satisfiable if the point v is moved vertically by a distance $d' > d(l^*, v^*)$. Note that Property 2 is not true if the vertical width is replaced by the width of the polygon. Such a geometrical interpretation could not be done with this measure.

Thus, the measure we propose seems to be a good candidate to estimate the distance between a discrete curve and a discrete circle. Nevertheless, this geometrical interpretation supposes that all the constraints we deal with are independent one from another, such that moving one constraint does not affect the others. But this is not true for the data we deal with since each constraint (i.e. bisector) is computed from two pixels (one in $\mathcal{S}$, the other in $\mathcal{T}$) and one pixel is used in the definition of more that one bisector. We see that the general framework that should be studied to get a precise estimation of the circularity by this method is a very difficult combinatorial problem. However, in the next section, experimental results show that the measure we propose gives good results for some classes of objects.

3.2 Algorithm and complexity

Algorithm 1, summarized hereafter, has been detailed in section 2.

Algorithm 1: The algorithm which computes our new circularity measure

Input: a closed discrete curve
Output: a circularity measure
1 Compute $\mathcal{S}$ and $\mathcal{T}$ from the input;
2 Compute parameters (a, b, c) of half-planes $\mathcal{H}(s, t)$;
3 Compute $\mathcal{K}^+$ and $\mathcal{K}^-$ according to the sign of b;
4 $\mathcal{L}^{+^*}$ = ConvexHull(Dual($\mathcal{K}^+$));
5 $\mathcal{L}^{-^*}$ = ConvexHull(Dual($\mathcal{K}^-$));
6 **return** VerticalWidth($\mathcal{L}^{+^*} \bigcap \mathcal{L}^{-^*}$);

The overall complexity of the algorithm is the same that the one of the recognition algorithm [CGRT04], that is $\mathcal{O}(n^{4/3}.log(n))$. Indeed, step 1 is in $\mathcal{O}(n)$ thanks to the arithmetical approach described in [CGRT04]. Steps 2 and 3 are in

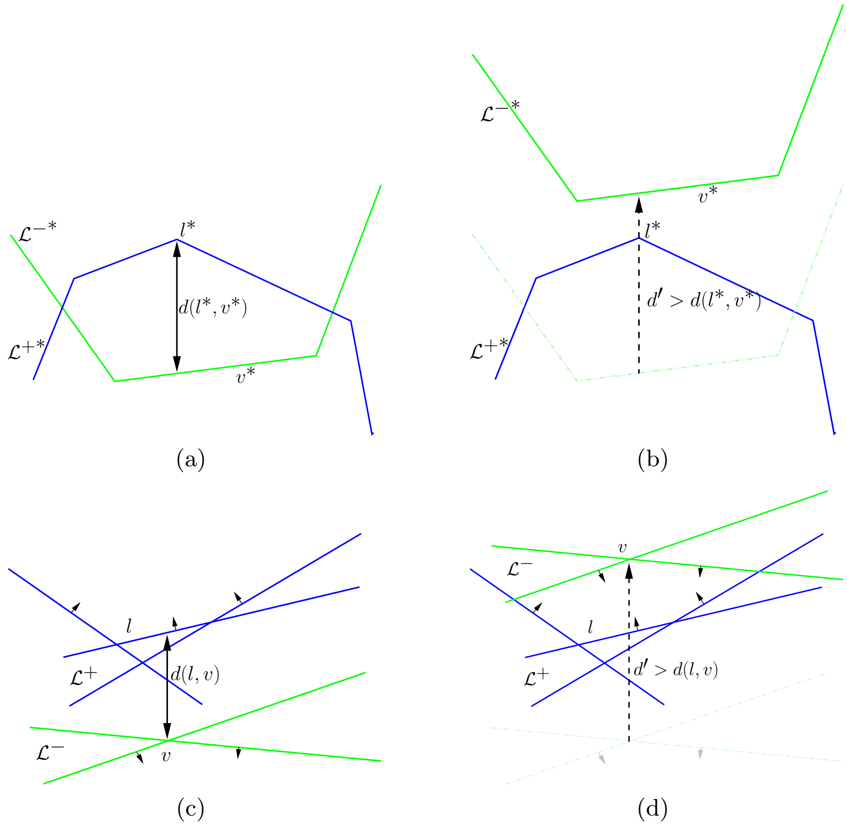


Fig. 4. Geometrical interpretation of the vertical width of $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*}$: $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*}$ becomes empty when v^* is moved by d', while in the primal space, the constraints become satisfiable.

$\mathcal{O}(n'^2)$ where $n' = card(\mathcal{S}) = card(\mathcal{T})$. Steps 4 and 5 are in $\mathcal{O}(n'^2 log(n'))$ using classical convex hull computation algorithms. Steps 6 is in $\mathcal{O}(log(n''))$ where n'' is the number of vertices of $\mathcal{L}^{+*} \bigcap \mathcal{L}^{-*}$. Knowing that n' is bounded by $\mathcal{O}(n^{2/3})$ ([CGRT04]) and $n'' < n'$, we can conclude that the overall complexity of the algorithm is $\mathcal{O}(n^{4/3}.log(n))$.

4 Experiments

By definition, our new circularity measure is minimum and equals 0 for all discrete circles whatever its center or its radius (see section 3). In this section we study the behavior of this measure with other classes of objects, either smooth objects that are not circles, such as ellipses and polygons or noisy objects.

4.1 Smooth objects

First, we generated hundreds of discrete ellipses (OBQ discretization) with various parameters : a (resp. b), small (resp. great) semi-axis, θ, the angle between the main axis of the ellipse and the x-coordinate axis, O_x and O_y the coordinates of the ellipse center. Figure 5 and Figure 6 show that circularity linearly inscreases with size whereas it exponentially inscreases with eccentricity. This behavior recalls the true circularity measure apart from the fact that the proposed measure is not yet normalized.

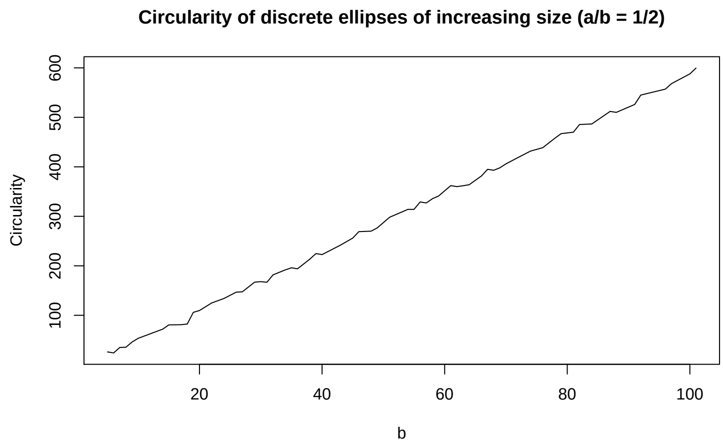


Fig. 5. One hundred of discrete ellipses were generated according to the following rules: $O(0,0)$, $\theta = 0$, $a/b = 1/2$ and b is ranging from 5 to 105. Circularity is plotted against b, the size of the ellipses.

Next, we generated fifty regular polygons of fixed perimeter. Their number of sides is ranging from 3 to 53, whereas their perimeter is approximatively equal to 1325 (the unit is pixel). In Figure 7, circularity decreases with the number of sides and converges towards 0. The bigger the number of sides, the more the polygons look like a circle and the more circularity is close to 0. Note that the artefacts are related to the parity of the number of sides.

4.2 Noisy objects

Finally, we generated hundreds of noisy circles. In order to study the impact of the amount of noise onto circularity, we implemented a degradation model very close to the one in [KHB$^+$00]. This model was proposed and validated in the context of document analysis and assume that: (i) the probability to flip a pixel (*i.e.*, label 'object' or '1' a pixel labelled 'background' or '0' and conversely)

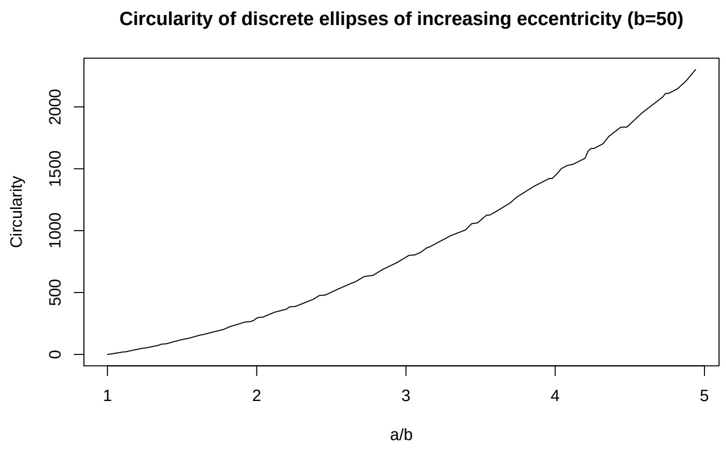


Fig. 6. One hundred of discrete ellipses were generated according to the following rules: $O(0,0)$, $\theta = 0$, $b = 50$ and a is ranging from 10 to 50. Circularity is plotted against a/b, the eccentricity of the ellipses.

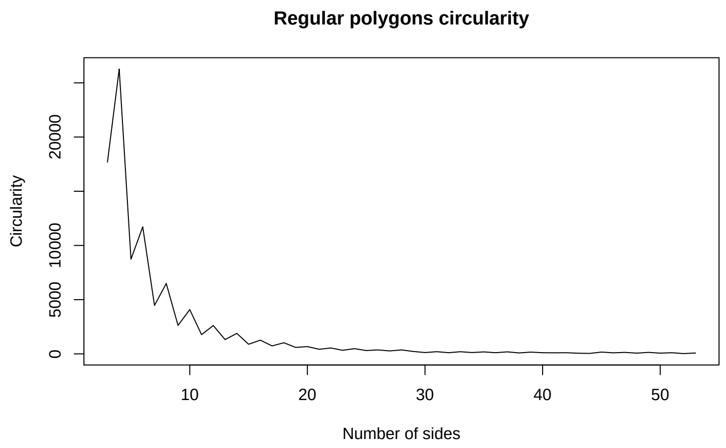


Fig. 7. 50 regular polygons, the perimeter of which is approximatively equals to 1325. Circularity is plotted against the number of sides.

depends of its distance to the nearest pixel of the complement set and (ii) other acquisition defaults, such as blur, may be simulated with a morphological closing. Thus, in practice:

- we perform a squared euclidean distance transform (see [Hir96]),
- we process each pixel according to formula 1 which is a simplified version of the one of [KHB$^+$00]:

$$p(0|1) = p(1|0) = \exp\left(-\frac{d^2}{\alpha}\right) \tag{1}$$

  where d^2 is the squared euclidean distance of the pixel to process to the nearest pixel of the complement set and α is a parameter which controls the amount of noise.
- we apply a morphological closing with a circular structuring element, the radius of which is 5, in order to make the object connected again.

Figure 8 gives some results of the algorithm applied to a discrete disk.

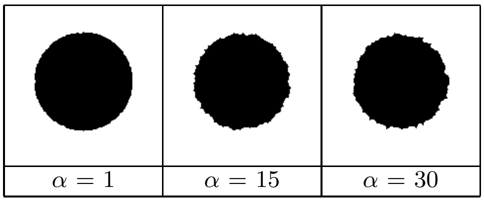


Fig. 8. Three disks, the radius of which is 50. The added noise depends of parameter alpha, according to the degradation model of [KHB$^+$00]. The circles we want to process are the 8-connected boundaries of the disks.

In Figure 9, circularity increases with the amount of noise, but not in a smooth way, contrary to previous plots. Globally, the noisier the circle, the more it looks different from a circle and the higher circularity is. However, we have to keep in mind that both noise and circularity definitions are subjectives.

5 Conclusion and perspectives

In this paper, a new circularity measure for discrete objects is introduced and an algorithm to compute this measure is presented (section 3).

An important property is fulfilled: the measure equals 0 for all discrete circles and is strictly positive for all discrete objects which are not discrete circles. Experiments show good and encouraging results (section 4). However, the measure proposed need to be normalized (Figure 5), in order to be scale invariant with discrete objects such as discrete ellipses. In addition, the measure proposed is not rotation invariant and it is one of our future works.

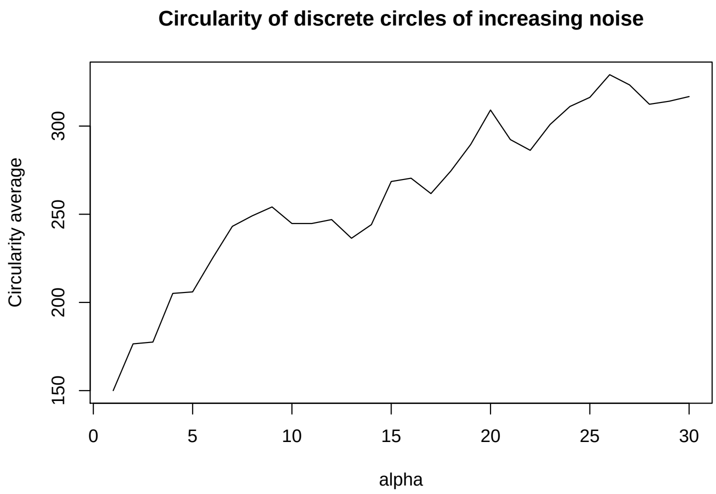


Fig. 9. Nine sets of circles that are more and more noisy have been generated. Parameter alpha ranging from 1 to 30 controls the amount of noise (see Figure.8). Average circularity measure is plotted against parameter alpha.

The problem of computing a consistent circularity measure is formulated in terms of sets of inequalities and we use a dual space to represent these sets. In this dual space, a measure of a polygon is computed to estimate the distance from a discrete curve to a discrete circle. Very tight links between this method and the field of linear programming need to be studied, especially the case of sets of constraints insatisfiable. This method is very general and could be applied to other objects.

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