Direct Spherical Harmonics Transform of a Triangulated Mesh

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Abstract

Spherical harmonics transform plays an important role in research in shape description. Current methods to compute the spherical harmonics decomposition of the characteristic function of the intersection of a polyhedral solid with a sphere involve expensive voxelization, and are prone to numerical errors associated with the size of the voxels. This paper describes a fast and accurate technique for computing spherical harmonics coefficients directly from the description of the mesh. The algorithm runs in linear time $O(pn)$, where n is the number of triangles of the mesh and p is the number of terms calculated, which is roughly linear.

1 Introduction

If M is a closed embedded surface in 3D space then there is a solid volume V bounded by M. Fixing a point P, one can consider the intersection of V with a sphere S_r of radius r around P. In general, we choose P at the centroid of the object. The intersection $M_r = V \cap S_r$ is a region of the sphere. The characteristic function χ_r of this region (1 for points inside, 0 for those outside) can be approximated by a sum of spherical harmonics $Y_l^m(\theta, \varphi)$ in the spherical coordinates. Theoretically, the expansion of χ_r in terms of spherical harmonics is written as:

$$\chi_r(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} c_{lm}^r Y_l^m(\theta, \varphi) \tag{1}$$

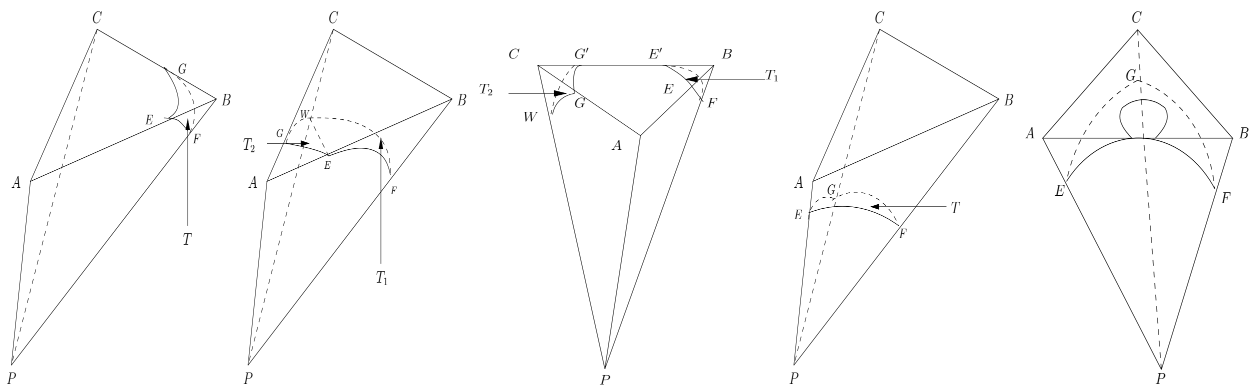
This expansion corresponds to a frequency-based decomposition of χ_r. In practice, since higher order coefficients c_{lm}^r correspond to finer details of the objects (maybe noise), we limit this summation to a bandwidth denoted as bw :

$$\chi_r(\theta, \varphi) \approx \sum_{l=0}^{bw} \sum_{|m| \leq l} c_{lm}^r Y_l^m(\theta, \varphi) \tag{2}$$

A collection of such approximations of the characteristic function χ_r, for several values of r, has proved to be useful as shape-descriptors for searching and identifying 3D objects [3, 9].

In this paper, we show how to compute the coefficients of the spherical harmonics representation of χ_r quickly and accurately when the surface M is given by a polyhedral mesh. The computation of each coefficient c_{lm}^r has an $O(n)$ complexity, where n is the number of triangles of the mesh. To calculate the spherical harmonics representation of χ_r, there is also a dependence on the number of terms calculated, which is roughly linear.

Standard algorithms [3, 6, 8] compute a voxelization of V and then use this discrete approximation to find the coefficients. The discretization introduces errors in the integrations needed to compute the harmonic coefficients. Furthermore, when the harmonics have high degree, the grid-spacing necessary to get accurate integration becomes smaller and smaller, and this, too, can add to the complexity. In contrast, our method does not require this discrete approximation of V to calculate the integration. The numerical quality of the voxelized methods can approximate ours when the voxel grid is chosen small enough. Mesh voxelization can be computed in a time that is approximately linear in the number of voxels that meet the mesh [5]. The integration of the function over these voxels is then linear in the number of voxels. Our algorithm is linear in the number of mesh triangles. Thus if the voxelization generates approximately a constant number of voxels per triangle, the two methods have identical asymptotic running times.



(a) A and C are inside the sphere S_r. EG is the only non geodesic arc.

(b) A is inside the sphere S_r. EG is the only non geodesic arc.

(c) A is inside the sphere S_r and S_r intersects twice the edge BC. EE' and GG' are the only non geodesic arcs.

(d) A, B and C are outside the sphere. All arcs are geodesics.

(e) A, B and C are outside the sphere. However ABC intersects the sphere.

Figure 1: intersections of the sphere S_r and a tetrahedron H.

2 Decomposition

The volume V can be defined as a union of tetrahedra, one from each triangle of the mesh to the point P. The triangles of the mesh are assumed to be oriented consistently, i.e., for two neighboring triangles, the shared edge has two different directions. If the normal of a triangle and P are in the opposite side with respect to the triangle, the corresponding tetrahedron is said to be positive, and negative otherwise.

Let $\{H_k, k = 1, \ldots, n\}$ denote the set of signed tetrahedra. We split $\{H_k\}$ into two subsets $\{H_i^+\}$ and $\{H_j^-\}$ according to the signs of the tetrahedra. The signed volume enclosed by the triangulated mesh can be represented as :

$$V = \left(\bigcup_i H_i^+\right) - \left(\bigcup_j H_j^-\right) \tag{3}$$

More formally, this means that a point is defined to be in V if the sum of the signs of all tetrahedra that it occupies is positive. Zhang and Chen [11] have shown that this decomposition can be used to calculate global volumetric moments on V as a sum of elementary volumetric moments computed on each tetrahedron of the decomposition.

$$moment(V) = \sum_k sign(H_k) moment(H_k) \tag{4}$$

We extend this property to directly compute the spherical harmonics transform of the triangulated mesh M. To compute the spherical harmonics coefficients of χ_r, we observe that the set $M_r = V \cap S_r$ is a signed union of spherical triangles $\{T_k, \quad k = 1, ..., s_n\}$ arising from the intersections of the sphere S_r with the tetrahedra of V. Let χ_{T_k} denote the characteristic function of T_k. Therefore χ_r can be represented as :

$$\chi_r(\theta, \varphi) = \sum_k sign(T_k) \chi_{T_k}(\theta, \varphi) \tag{5}$$

and

$$c_{l,m}^r = \sum_k sign(T_k) c_{l,m}^r|_{T_k} \tag{6}$$

The decomposition of M_r into a signed union of spherical triangles is obtained from the intersection of S_r with the set of signed tetrahedra $\{H_k\}$. There are four general cases for the intersection of S_r and the tetrahedron $H = PABC$ (recall that P is the center of the sphere).

- If A, B and C are inside S_r, then there is no intersection.
- If one of A, B and C is outside S_r, then S_r intersects the edges of H at three points forming a spherical triangle T (figure 1(a)).

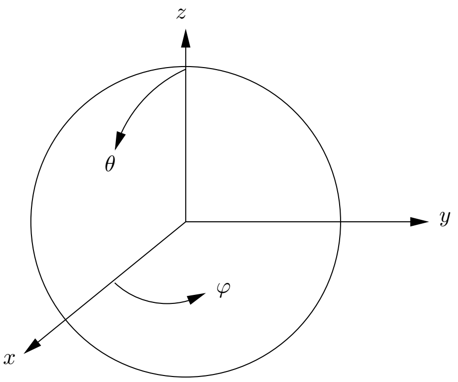


Figure 2: direction of θ and φ in the spherical coordinate

- If two of A, B and C are outside S_r, then S_r intersects the edges of H at four or six points forming a union of two spherical triangles $T_1 \cup T_2$ (figure 1(b) and 1(c)).
- If A, B and C are all outside S_r, then S_r intersects the edges of H at three points forming a spherical triangle T (figure 1(d)). An interior part of the triangle ABC may be inside S_r (figure 1(e)). Then the spherical triangle T does not entirely lie inside H. To deal with that case, let Q be a point on ABC lying inside S_r. Then the tetrahedron H can be seen as the union of the tetrahedra $PABQ$, $PBCQ$ and $PCAQ$ which can be treated as in the second case (figure 1(b) or 1(c)). Note that those tetrahedra raise spherical triangles with at most one non geodesic arc for each.

The sign of a spherical triangle is inherited from the tetrahedron it comes from. We compute the spherical harmonics coefficients for the characteristic function of a single spherical triangle EFG, and then sum the results. The details of the summation process are described in the next section.

3 The coefficients for a single spherical triangle

In this section, we describe the calculation of the harmonic coefficients over a single spherical triangle T. We describe it for general spherical triangles. To find the spherical harmonics coefficient $c^r_{l,m}|_T$ for the characteristic function χ_T, we must compute:

$$c^r_{l,m}|_T = \int\int_{\mathbb{S}^2} \chi_T(\theta,\varphi)\overline{Y}_l^m(\theta,\varphi)\sin(\theta)d\theta d\varphi \tag{7}$$

Y_l^m is the spherical harmonics of degree l and order m where $l \geq 0$ and $|m| \leq l$, $\theta \in [0 \ \ \pi]$ is the polar angle from the z-axis, and $\varphi \in [0 \ \ 2\pi[$ is the azimuthal angle in the xy-plane from the x-axis (figure 2). The expression of $c^r_{l,m}|_T$ is the projection of χ_T on Y_l^m since the spherical harmonics are a set of orthogonal functions. Recall that $Y_l^m(\theta,\varphi)$ is defined as :

$$Y_l^m(\theta,\varphi) = k_{lm}P_l^m(\cos\theta)e^{im\varphi} \tag{8}$$

k_{lm} is a constant depending on l and m, and P_l^m is the associated Legendre Polynomial [10]. For more details about spherical harmonics see [1, 4].

The expression to be integrated is nonzero only for points in EFG, so $c^r_{l,m}|_T$ simplifies to

$$c^r_{l,m}|_T = \int\int_T \overline{Y}_l^m(\theta,\varphi)\sin(\theta)d\theta d\varphi \tag{9}$$

The harmonic coefficients $c^r_{l,m}|_T$ are complex. They are related to each other by the following relation:

$$c^r_{l,-m}|_T = (-1)^m\overline{c^r_{l,m}|_T} \tag{10}$$

This allows one to do half as much work as would be otherwise necessary.

Although directly integrating the spherical harmonics over a single spherical triangle might appear simple, symbolic packages offered by Mathlab or Mathematica do not permit, as far as we know, to calculate the integration expressed in equation 9, because of the non linear relation between θ and φ on the domain boundaries, (see [7] for more details). We therefore prefer a general numerical approach, which we now describe.

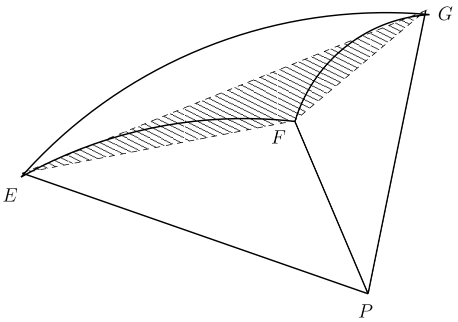


Figure 3: a spherical triangle and its corresponding euclidean triangle.

4 Numerical Estimation

A spherical triangle EFG bounded by geodesic arcs (and which is smaller than its complementary on the sphere) can be parametrized by the euclidean triangle sharing the same vertices (figure 3). Therefore, the radial projection from EFG to the corresponding standard triangle can be used to parametrize the integration over EFG. The function we want to integrate is :

$$\chi(\theta,\varphi)\overline{Y}_l^m(\theta,\varphi) \tag{11}$$

where χ is 1 for each point of EFG and 0 otherwise. In fact, each point Q on the euclidean triangle EFG can be written as:

$$Q = \lambda E + \beta F + (1-\lambda-\beta)G \qquad 0 \le \lambda \le 1 \quad 0 \le \beta \le 1 \quad 0 \le \lambda + \beta \le 1 \tag{12}$$

We can parametrize (θ,φ) by (λ,β) using the conversion from the cartesian to spherical coordinates:

$$\theta(\lambda,\beta) = \arctan\frac{\sqrt{x(\lambda,\beta)^2 + y(\lambda,\beta)^2}}{z(\lambda,\beta)} \tag{13}$$

$$\varphi(\lambda,\beta) = \arctan\frac{y(\lambda,\beta)}{x(\lambda,\beta)} \tag{14}$$

Therefore, the integration over a spherical triangle bounded by geodesic arcs becomes:

$$c^r_{l,m}|_T = \int_0^1\int_0^1 \chi(\theta(\lambda,\beta),\varphi(\lambda,\beta))\overline{Y}_l^m(\theta(\lambda,\beta),\varphi(\lambda,\beta))\sin\theta(\lambda,\beta) J\left[\frac{\theta,\varphi}{\lambda,\beta}\right] d\lambda d\beta \tag{15}$$

where $J\left[\frac{\theta,\varphi}{\lambda,\beta}\right]$ is the Jacobian of (θ,φ) with respect to (λ,β). Equation (15) can be evaluated using one of numerical integrations implemented in any scientific library. The examples presented in this article were evaluated using the plain Monte Carlo integration implemented in the GNU Scientific Library, GSL [2] with 10^3 as the size of the iteration space.

The numerical estimation has the following advantages:

- it need not partition the spherical triangle,
- it maintains implicitly the dependency between θ and φ,
- it can compute high order harmonic coefficients,

The numerical estimation described in this section can handle the case of the spherical triangle whose one boundary is not geodesic, see Figures 1(a), 1(b) and 1(c), but also spherical triangle that miss an interior part, see Figure 1(e), without splitting it into smaller ones. In fact, the integrand, in equation 15, evaluates to 0 for the points that do not lie inside the tetrahedron. The test of inside/outside the tetrahedron is what is done by the function χ.

An overview of our method is summarized by the following algorithm :

Algorithm 1 overview of our algorithm

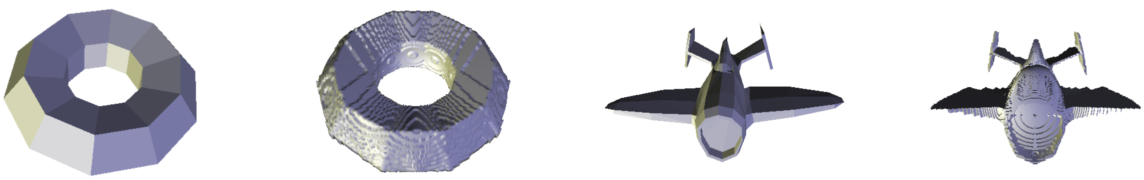
input: a triangulated mesh M, a sphere S_r with radius r and centered at P
output: $c^r_{l,m}$
$T \leftarrow$ the set of triangles of M
$H \leftarrow$ the set of signed tetrahedra induced by P and T
$c^r_{l,m} \leftarrow 0$
for all $h \in H$ **do**
  $\Gamma \leftarrow$ the set of the spherical triangles resulting from $h \cap S_r$, Figure 1
  for all $t \in \Gamma$ **do**
    $c^r_{l,m} \leftarrow c^r_{l,m} + sign(h) \times c^r_{l,m}|_t$, {where $c^r_{l,m}|_t$ is evaluated using equation 15}
  end for
end for
return $c^r_{l,m}$

5 Examples and Discussion

We have tested our method with different kinds of objects. Indeed, the quality of the spherical harmonics description of the object depends both on the number of spheres and on the bandwidth bw. Figure 4 and 5 show a reconstruction of different kind of models using different spheres radii and bandwidths.

To simplify the calculations, we normalize the models and translate them so that the centers of masses coincide with the origin. The spheres are chosen to be equispaced. We can exploit an interesting property offered by our decomposition. When the radius of the sphere changes and the intersection of the tetrahedron H and the sphere is always as depicted in figure 1(d) then the spherical harmonics coefficients associated to H remain the same. The source code available online does not include this optimization, since we want to make it more simple for the reader to comprehend the main algorithm.

The integrations performed over a single triangle are independent of the other triangles. Therefore the calculation can be parallelized without any overlapping problem.



(a) original (b) 50 spheres (c) original (d) 100 spheres

Figure 4: reconstruction of a polygonized torus and aircraft using $bw = 64$

(a) original (b) 100 spheres (c) original (d) 100 spheres

Figure 5: reconstruction of a horse and Isis statue using $bw = 64$

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References

[1] W. E. Byerly. *Spherical Harmonics*, chapter 6, pages 195–218. New York: Dover, 1959. An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics.

[2] M. Galassi et al. *GNU Scientific Library Reference Manual (2nd Ed.)*. http://www.gnu.org/software/gsl/.

[3] T. Funkhouser, P. Min, M. Kazhdan, J. Chen, A. Halderman, D. Dobkin, and D. Jacobs. A search engine for 3d models. *ACM Transactions on Graphics*, 22(1):83–105, January 2003.

[4] E. W. Hobson. *The Theory of Spherical and Ellipsoidal Harmonics*. New York: Chelsea, 1955.

[5] A. Kaufman and E. Shimony. 3d scan-conversion algorithms for voxel-based graphics. In *SI3D '86: Proceedings of the 1986 workshop on Interactive 3D graphics*, pages 45–75, New York, NY, USA, 1987. ACM Press.

[6] M. Kazhdan, T. Funkhouser, and S. Rusinkiewicz. Rotation invariant spherical harmonic representation of 3d shape descriptors. In *SGP '03: Proceedings of the Eurographics/ACM SIGGRAPH symposium on Geometry processing*, pages 156–164. Eurographics Association, 2003.

[7] M. Mousa, R. Chaine, and S. Akkouche. Direct spherical harmonic transform of a triangulated mesh. Technical Report RR-LIRIS-2005-015, LIRIS : Lyon Research Center for Images and Intelligent Information Systems, 2005.

[8] D. Saupe and D. V. Vranic. 3d model retrieval with spherical harmonics and moments. In *Proceedings of the 23rd DAGM-Symposium on Pattern Recognition*, pages 392–397. Springer-Verlag, September 2001.

[9] D. V. Vranic and D. Saupe. Description of 3d-shape using a complex function on the sphere. In *ICME '02: Proceedings of the IEEE International Conference on Multimedia and Expo*, pages 177–180, 2002.

[10] Eric W. Weisstein. *Legendre Polynomial*. From MathWorld : A Wolfram Web Resource. http://mathworld.wolfram.com/LegendrePolynomial.html.

[11] C. Zhang and T. Chen. Efficient feature extraction for 2d/3d objects in mesh representation. In *ICIP '01: Proceedings of the International Conference on Image Processing*, pages 935–938, October 2001.