Digital Planarity - A Review

Valentin Brimkov,[a] David Coeurjolly,[b] and Reinhard Klette[c]

[a]*State University of New York, Fredonia, NY 14063, USA*

[b]*LIRIS, CNRS FRE 2672, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne, France*

[c]*CITR, University of Auckland, Tamaki campus, Auckland 1005, New Zealand*

Abstract

Digital planarity is defined by digitizing Euclidean planes in the three-dimensional digital space of voxels; voxels are given either in the grid point or the grid cube model. The paper summarizes results (also including most of the proofs) about different aspects of digital planarity, such as self-similarity, supporting or separating Euclidean planes, characterizations in arithmetic geometry, periodicity, connectivity, and algorithmic solutions. The paper provides a uniform presentation, which further extends and details a recent book chapter in (Klette and Rosenfeld 2004).

Key words: digital planes, digital geometry, arithmetic geometry, 2D array periodicity, Sturmian words, geometric algorithms
1991 MSC: 52C99, 62H35, 65D18, 68U05

1 Introduction

In this paper we review various concepts and results of digital planarity and exhibit relations of the subject to other disciplines. Some of the considered matters are partially familiar from studies on digital straightness. However, digital planarity issues appear to be more challenging, due to further significant applications in pattern recognition and volume modeling, and to certain theoretical obstacles caused by the higher dimension. We conform to traditional terminology adopted in digital geometry. For some basic notions the reader is referred to (Klette and Rosenfeld 2004).

A plane in $\mathbb{R}^3$ whose z-coefficient is not 0, is defined by an expression of the form

$$\Gamma(\alpha_1, \alpha_2, \beta) = \{(x, y, z) \in \mathbb{R}^3 : \ z = \alpha_1 x + \alpha_2 y + \beta\}$$

where $\alpha_1, \alpha_2, \beta \in \mathbb{R}$. The symmetry of the grid allows us to assume $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$ throughout this paper; we also assume $0 \leq \beta < 1$ for convenience using an argument similar to that for digital straight lines.

A digital plane can be defined from such a plane in Euclidean space $\mathbb{R}^3$ by applying a specific model of digitization. It is common to choose either outer 3D Jordan digitization (also known as supercover digitization) which assigns all grid cubes having a non-empty intersection with the given plane, or 3D grid-line intersection digitization which assigns all those grid points which are nearest to an intersection of the plane with a grid line; or simply by applying the floor or ceiling function to the coordinates of the points in $\Gamma(\alpha_1, \alpha_2, \beta)$.

We start with the grid-point model and assume grid-line intersection digitization in this case. Under the above assumptions for $\alpha_1, \alpha_2, \beta$, it is sufficient to consider only intersections with grid lines parallel to the z-axis. Let $\Gamma(\alpha_1, \alpha_2, \beta)$ intersect the vertical grid line $(x = m, y = n)$ at $p_{m,n}$ where $m, n \geq 0$. Let $(m, n, I_{m,n})$ be the grid point closest to $p_{m,n}$. Then a *digital plane quadrant* is a set of grid points which is defined as follows:

$$I_{\alpha_1,\alpha_2,\beta} = \{(m, n, I_{m,n}) : m, n \geq 0 \wedge I_{m,n} = \lfloor \alpha_1 m + \alpha_2 n + \beta + 0.5 \rfloor\}$$

If there are two closest grid points, then we choose the upper one. If m, n are not required to be nonnegative, we have a *digital plane*. The set $I_{\alpha_1,\alpha_2,\beta}$ uniquely determines both the slopes α_1 and α_2 and the intercept β if α_1 or α_2 is irrational. If both α_1 and α_2 are rational, $I_{\alpha_1,\alpha_2,\beta}$ uniquely determines α_1 and α_2, but determines β only up to an interval. This can be proved by a 3D generalization of the proof of Bruckstein's Theorem (Bruckstein 1991) as stated in (Rosenfeld and Klette 2001).

In analogy with the chain codes for digital curves (Freeman 1961), and following (Brimkov 2002), we define *step codes*, starting with $i_{\alpha_1,\alpha_2,\beta}(0,0) = I_{0,0} \in \{0, 1\}$, as follows:

$$i_{\alpha_1,\alpha_2,\beta}(0, n+1) = I_{0,n+1} - I_{0,n} = \begin{cases} 0 & \text{if } I_{0,n+1} = I_{0,n} \\ 1 & \text{if } I_{0,n+1} = I_{0,n} + 1 \end{cases} \quad \text{for } n \geq 0$$

$$i_{\alpha_1,\alpha_2,\beta}(m+1, 0) = I_{m+1,0} - I_{m+1,0} = \begin{cases} 0 & \text{if } I_{m+1,0} = I_{m,0} \\ 1 & \text{if } I_{m+1,0} = I_{m,0} + 1 \end{cases} \quad \text{for } m \geq 0$$

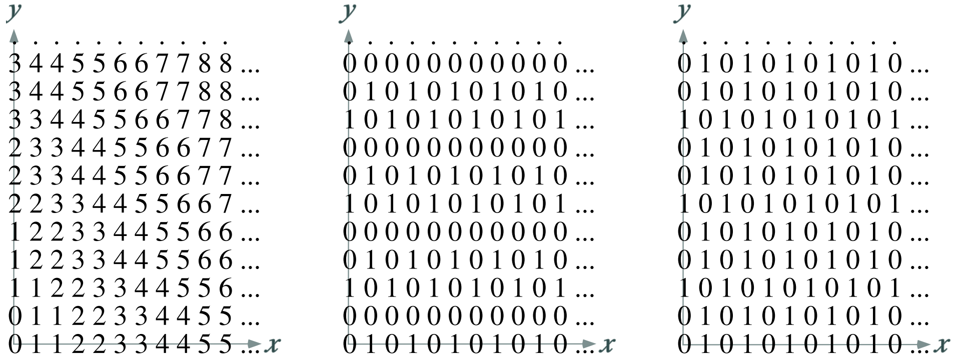


Fig. 1. (Brimkov 2002) Left: $I_{\frac{1}{2},\frac{1}{3},0}(m, n)$. Middle: $i^{(r)}_{\frac{1}{2},\frac{1}{3},0}(m, n)$. Right: $i^{(c)}_{\frac{1}{2},\frac{1}{3},0}(m, n)$.

In addition to these "initial values," we also define column-wise step codes

$$i^{(c)}_{\alpha_1,\alpha_2,\beta}(m,n+1) = I_{m,n+1} - I_{m,n} = \begin{cases} 0 & \text{if } I_{m,n+1} = I_{m,n} \\ 1 & \text{if } I_{m,n+1} = I_{m,n} + 1 \end{cases} \quad \text{for } m \geq 1$$

and row-wise step codes

$$i^{(r)}_{\alpha_1,\alpha_2,\beta}(m+1,n) = I_{m+1,n} - I_{m,n} = \begin{cases} 0 & \text{if } I_{m+1,n} = I_{m,n} \\ 1 & \text{if } I_{m+1,n} = I_{m,n} + 1 \end{cases} \quad \text{for } n \geq 1$$

Values in the 0th row and 0th column are used in both the column-wise and row-wise step codes; see Figure 1. Assumptions $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$ guarantee that codes 0 and 1 are sufficient. Based on the additional assumption $\alpha_1 \leq \alpha_2$, we will only use row-wise step codes in the sequel, and will omit the superscript (r).

Definition 1 (Brimkov 2002) $i_{\alpha_1,\alpha_2,\beta} = \{(m, n, i_{\alpha_1,\alpha_2,\beta}(m, n)) : m, n \geq 0\}$ is a *step code of a digital plane quadrant* (in the grid point model), or, for short, a *quadrant step code*, with slopes α_1 and α_2 and intercept β.

If we do not require m, n to be nonnegative integers, we obtain a *step code of a digital plane*. For short, we call it a *plane step code*.

Digital planes and plane quadrants have analogous properties, as plane and quadrant step codes do. To simplify our notation, we will use $I_{\alpha_1,\alpha_2,\beta}$ to denote both digital planes and plane quadrants, and $i_{\alpha_1,\alpha_2,\beta}$ for plane or quadrant step codes.

For $D \subseteq \mathbb{R}^2$, let

$$i^D_{\alpha_1,\alpha_2,\beta} = \{(m, n, i_{\alpha_1,\alpha_2,\beta}(m, n)) : (m, n) \in D \cap \mathbb{Z}^2\}$$

If α_1 or α_2 is irrational, then we speak about an *irrational digital plane* or an *irrational plane step code*, respectively; otherwise it is a *rational digital plane*, resp. a *rational plane step code*.

(Lunnon and Pleasants 1992) shows that rational digital straight lines are translation-equivalent if they have identical slopes. Rational digital planes with identical slopes are also translation-equivalent; see (Brimkov and Barneva 2003). This implies that translation-invariant properties of rational digital planes are independent of intercepts; the translation equivalence classes of all rational digital planes can be uniquely identified by I_{α_1,α_2} or i_{α_1,α_2}.

A digital plane is a special case of a digital surface. An early definition of a digital surface is the following.

Definition 2 (Kim 1984) A 26-connected set $S \subseteq \mathbb{Z}^3$ is called a *digital surface* iff each $p = (i, j, k) \in S$ has at most two 8-adjacent grid points in at least two of the sets $\{(y, z) : (x, y, z) \in S \wedge x = i\}$, $\{(x, z) : (x, y, z) \in S \wedge y = j\}$, or $\{(x, y) : (x, y, z) \in S \wedge z = k\}$; if it has two, then they are not mutually 8-adjacent; and if p has in one of these sets, say, $\{(x, y) : (x, y, z) \in S \wedge z = k\}$, more than two 8-adjacent grid points, or two 8-adjacent grid points that are mutually 8-adjacent, then $(i, j, k-1)$ and $(i, j, k+1)$ are not in S.

Theorem 3 *(Kim 1984) A digital plane is an unbounded digital surface.*

PROOF. Let $p = (i, j, k)$ be a point of digital plane $I_{\alpha_1,\alpha_2,\beta}$, and consider $I_{\alpha_1,\alpha_2,\beta} \cap \{(x, y, z) \in S : x = i\}$. Let $p' = (i, j-1, k')$ and $p'' = (i, j+1, k'')$ be the only two points of $I_{\alpha_1,\alpha_2,\beta}$ on the vertical lines $x = i$ and $y = j-1$ and $x = i$ and $y = j+1$, respectively. Since $\alpha_1 \leq \alpha_2$, we have $0 \leq |k - k'|, |k - k''| \leq 1$. Thus $(j-1, k')$ and $(j+1, k'')$ are the only two points defined by p and $x = i$, which are 8-adjacent to (j, k), but not mutually 8-adjacent. Similarly, p and $y = j$ define only two 8-adjacent points in $I_{\alpha_1,\alpha_2,\beta} \cap \{(x, y, z) \in S : y = j\}$, which are not mutually 8-adjacent. In $I_{\alpha_1,\alpha_2,\beta} \cap \{(x, y, z) \in S : z = k\}$, p and $z = k$ may define more than two 8-adjacent points. However, $(i, j, k-1)$ and $(i, j, k+1)$ are not both in $I_{\alpha_1,\alpha_2,\beta}$ since $p = (i, j, k)$ is the only point of $I_{\alpha_1,\alpha_2,\beta}$ on the vertical grid line $x = i$, $y = j$. Thus it follows that $I_{\alpha_1,\alpha_2,\beta}$ is a digital surface. □

A grid point $p = (i, j, k)$ of a digital surface S is called a *border point* of S iff it has only one 26-neighbor in $\{(x, y, z) \in S : x = i\}$, $\{(x, y, z) \in S : y = j\}$, or $\{(x, y, z) \in S : z = k\}$. p is called an *inner point* of S iff it is not a border point. A *simple digital surface* is a digital surface that has no border points; it can be either unbounded or bounded *hole-free simple digital surface.* A *digital surface patch* is a finite digital surface whose border points are 26-connected.

Corollary 4 *Let $D \subset \mathbb{Z}^2$ be a 4-region; then $I^D_{\alpha_1,\alpha_2,\beta}$ is a digital surface patch.*

Such a patch is called a *digital plane segment* (DPS), defined in the grid-point model with respect to grid-line intersection digitization.

For an alternative discussion of digital planarity in the grid cube model we can uniquely identify each grid point as being the centroid of a grid cube, and we can stay in this case with the previously introduced notations. This way, a *cellular digital plane* is defined by a digital plane in the grid point model. (Alternatively, a cellular digital plane could also be defined by outer Jordan digitization of a plane Γ. However, if Γ passes through a grid vertex or contains a grid edge, then outer Jordan digitization would produce "locally thicker" cellular planes).

The grid cell model also allows to introduce further notions in the context of digital planarity. Let $\mathbb{C}_3^{(m)}$ be the class of all m-dimensional grid cells in $\mathbb{R}^3$, for $m = 0, 1, 2, 3$. An *incidence grid* $\mathbb{C}_3$ is defined by these classes and the incidence relation between all cells. We have that two cells f_1 and f_2 are incident iff f_1 is a facet of f_2 or vice versa. For example, a single 1-cell (i.e., a grid edge in $\mathbb{C}_3^{(1)}$) is incident with exactly two 0-cells (i.e., grid vertices), one 1-cell (the grid edge itself), four 2-cells (i.e., grid squares), and four 3-cells (i.e., grid cubes).

Consider the union of all grid cubes contained in a cellular digital plane, within the Euclidean topology of $\mathbb{R}^3$. Its frontier consists of two "parallel layers" of *frontier faces*, which define an *upper* and a *lower digital frontier plane* in the incidence grid $\mathbb{C}_3$. Note that these are analogous to lower and upper digital lines defined in (Rosenfeld and Klette 2001). Upper and lower digital frontier plane share 0- and 1-cells, but not 2-cells.

Definition 5 A set $S \subset \mathbb{C}_3^{(2)}$ of 2-cells in the incidence grid is called a *digital plane of 2-cells* iff it is either an upper or a lower digital frontier plane defined by a cellular digital plane.

A finite 1-connected subset of a digital plane of 2-cells is called a *digital plane segment* (DPS) *of 2-cells* in the 3D incidence grid.

After this brief introduction of basic notions, in the following sections we review concepts and results related to digital planarity. The paper is structured as follows. In Section 2, we give some alternative definitions in terms of the chordal triangle property and evenness of surfaces. In Section 3, we characterize digital planes through supporting and separating planes, as well as in the framework of arithmetic geometry. In Section 4, we introduce height and remainder maps that are instrumental in studying periodicity and connectivity properties of digital planes. In Section 5, we review results on digital plane periodicity and self-similarity, while in Section 6 we address connectivity issues. In Section 7, we summarize a few algorithms for digital plane recognition, digital surface segmentation, and polyhedral surface generation. We conclude with some final remarks in Section 8.

2 Alternative Definitions

We use the Minkowski metric L_∞. If applied to $\mathbb{Z}^3$, it is identical to the grid point metric d_{26}.

Definition 6 (Kim and Rosenfeld 1982) $S \subseteq \mathbb{Z}^3$ is said to have the *chordal triangle property* iff for any $p_1, p_2, p_3 \in S$, every point on the triangle $p_1p_2p_3 \subset \mathbb{R}^3$ is at L_∞-distance < 1 from some point of S.

Obviously, a simple digital surface which satisfies the chordal property cannot be bounded.

Theorem 7 *(Kim 1984) A simple digital surface is a digital plane iff it has the chordal triangle property.*

The original proof is too long to be part of this review, therefore we only sketch it. First, Kim shows that, given a digital plane, there is a coordinate plane (the plane $z = 0$ according to our assumptions) such that the projection of the digital plane onto its grid points is a one-to-one and onto mapping (Kim 1984, Lemma 9). This lemma allows to reduce the dimension of the considered problem and to perform all considerations in such a coordinate plane rather than in 3D. Then the first implication (a digital plane has the chordal triangle property) can be easily derived (Kim 1984, Lemma 10). A key point is the existence of a Euclidean plane $\Gamma_{\alpha_1,\alpha_2,\beta}$ defining the given digital plane.

The proof of the converse proposition (a simple digital surface with the chordal triangle property is a digital plane) is more complicated. As a first step, it is shown that if a simple digital surface has the chordal triangle property, then there is a one-to-one and onto coordinate projection plane (Kim 1984, Lemma 11). Then the proof is completed by exhaustive analysis of different cases, conditioned by the distance between the supporting plane of a triangle and the points of the simple digital surface (Kim 1984, Lemma 12).

For any $p = (p_x, p_y, p_z) \in \mathbb{Z}^3$, let $p_{z=0} = (p_x, p_y, 0)$ be the projection of p onto the xy-plane.

Definition 8 $S \subseteq \mathbb{Z}^3$ is called *even* iff its projection onto the xy-plane $\{(x, y, 0) : (x, y) \in \mathbb{Z}^2\}$ is one-to-one, and for every quadruple (p, q, r, s) of points in S such that $p_{z=0} - q_{z=0} = r_{z=0} - s_{z=0}$, we have $|(p_z - q_z) - (r_z - s_z)| \leq 1$.

Defining evenness with respect to the xy-plane is consistent with our previous assumptions about digital planes. By requiring a one-to-one mapping onto the xy-plane, we consider only unbounded sets $S \subseteq \mathbb{Z}^3$ as being even. The following theorem does not make use of our general assumption that $\alpha_1 \leq \alpha_2$.

Theorem 9 *(Veelaert 1993) A simple digital surface is a digital plane iff it has the evenness property.*

Again we only sketch the original proof. As a first step, digital planarity is characterized in terms of linear programming: a set S of voxels is a subset of a digital plane if there exist $(\alpha_1, \alpha_2, \beta) \in [0,1] \times [0,1[$ such that

$$0 \leq \alpha_1 m + \alpha_2 n + \beta - I_{m,n} < 1$$

for all $(m, n, I_{m,n}) \in S$. Hence, to decide whether S is a subset of a digital plane, we have to solve a system of linear inequalities with unknowns α_1, α_2, and β. Given a voxel $(m, n, I_{m,n})$, an elementary convex open set associated with the given voxel is defined by two linear inequalities in three unknowns. S is a subset of a digital plane if the intersection of these elementary convex open sets is non-empty.

Next, one takes advantage of the following fundamental theorem of Helly: Let $\mathcal{F}$ be a finite family of $n+1$ or more convex subsets of $\mathbb{R}^n$. If every subfamily, consisting of $n+1$ sets of $\mathcal{F}$, has a non-empty intersection, then $\mathcal{F}$ has a non-empty intersection. Thus, in dimension 3, the system induced by the elementary convex sets has a solution if and only if each subsystem with four inequalities has a solution.

Finally, Veelaert proves that the evenness criterion can be used as a "Helly subsystem criterion." Note that the original result is valid for arbitrary dimensions. Moreover, it is shown that Kim's chordal triangle property is actually another Helly criterion.

(Veelaert 1994) also shows that for specific types of finite sets S of voxels (e.g., such that the projection onto the xy-plane is a rectangle), S is a subset of a digital plane iff S is even.

3 Supporting and Separating Planes

A *supporting plane* of a set $S \subseteq \mathbb{Z}^3$ divides $\mathbb{R}^3$ into two (open) half-spaces such that S is completely contained in the closure of one of them. For the next theorem note that any metric in $\mathbb{R}^3$ induces a Hausdorff distance between subsets of $\mathbb{R}^3$. We use the Minkowski metric L_∞.

Theorem 10 *(Kim 1984) $S \subseteq \mathbb{Z}^3$ is a digital plane iff it has a supporting plane Γ such that the L_∞-Hausdorff distance between S and Γ is less than 1.*

PROOF. Let $\Gamma(\alpha_1, \alpha_2, \beta)$ be a supporting plane for $S \subseteq \mathbb{Z}^3$. We assume without loss of generality that Γ is above S with respect to the z-axis. Let the L_∞-Hausdorff distance between S and Γ be < 1. Then the vertical distance from any point of S to Γ is < 1, as well. Denote by Γ' the plane obtained by translating Γ by a vector $(0, 0, -\frac{1}{2})^T$. Then Γ' is such that $S \subset I_{\alpha_1, \alpha_2, \beta - \frac{1}{2}}$ and so, S is a digital plane. Conversely, suppose that S is a digital plane.

Then there exists a plane $\Gamma(\alpha_1, \alpha_2, \beta)$ such that $S \subset I_{\alpha_1,\alpha_2,\beta}$. By definition of a digital plane, the vertical distance from any point of S to Γ is less than $\frac{1}{2}$. Let Γ' be the plane obtained by translation of Γ by a vector $(0, 0, \frac{1}{2})^T$. Then any point of S is below Γ' and the L_∞-Hausdorff distance between S and Γ' is < 1. Hence, Γ' is a supporting plane of S. □

In (Kim 1984) it was claimed that if $S \subseteq \mathbb{Z}^3$ is a (finite) DPS, then the points of S are at L_∞-Hausdorff distance < 1 from at least one Euclidean plane incident with one of the faces of the convex hull of S. Then one of these planes is a supporting plane in the sense of Theorem 10. However, (Debled-Rennesson 1995) gave a counter-example: for $D = [0, 6] \times [0, 7]$, the L_∞-Hausdorff distance between $I^D_{5/29,9/29,1/2}$ and any plane incident with one of the faces of the convex hull of $I^D_{5/29,9/29,1/2}$ is greater than 1.

Let $S \subset \mathbb{Z}^3$ and $S_{z+1} = \{(x, y, z+1) : (x, y, z) \in S\}$. A plane $\Gamma \subset \mathbb{R}^3$ *separates* the sets $S_1, S_2 \subset \mathbb{Z}^3$ iff S_1 and S_2 are in opposite open half-spaces defined by Γ.

Theorem 11 *(Stojmenović and Tosić 1991) A set $S \subset \mathbb{Z}^3$ is a subset of a digital plane iff there exists a plane that separates S from S_{z+1}.*

PROOF. We first suppose that S is a subset of a digital plane. Let $\Gamma(\alpha_1, \alpha_2, \beta)$ be the plane such that $S \subset I_{\alpha_1,\alpha_2,\beta}$ and Γ' the plane with parameters $(\alpha_1, \alpha_2, \beta+\frac{1}{2})$. We consider the points $r = (r_x, r_y, r_z) \in S$, $p = (r_x, r_y, p_z) \in \Gamma$, $p' = (r_x, r_y, p'_z) \in \Gamma'$, and $r_{z+1} = (r_x, r_y, r_z + 1) \in S_{z+1}$. From the definition of 3D grid-line intersection digitization and the definition of Γ', it follows that $p'_z - 1 < r_z \leq p'_z < r_z + 1$. Hence, the number p'_z "separates" the numbers r and r_{z+1}. Since this property is valid for every point of S, it follows that Γ' separates S from S_{z+1}, even if S is not finite. Conversely, let $\Gamma(\alpha_1, \alpha_2, \beta)$ be a separating plane for S and S_{z+1}. We consider $r \in S$, $r_{z+1} = (r_x, r_y, r_z + 1) \in S_{z+1}$ and $p = (r_x, r_y, p_z) \in \Gamma$. We have $r_z \leq p_z < r_z + 1$, i.e., $r_z - \frac{1}{2} \leq p_z - \frac{1}{2} < r_z + \frac{1}{2}$. Thus we obtain that the digital image of Γ' with parameters $(\alpha_1, \alpha_2, \beta - \frac{1}{2})$ is such that $S \subset I_{\alpha_1,\alpha_2,\beta-\frac{1}{2}}$. This means that S is a subset of a digital plane. □

Arithmetic geometry, as briefly indicated in (Forchhammer 1989) and developed in (Reveillès 1991), provides a uniform approach to the study of digitized hyperplanes in n dimensions. Basic definitions follow the general idea of specifying lower and upper supporting planes. We discuss here the three-dimensional case. Let a, b, c be relatively prime integers and let μ and $\omega \geq 0$ be integers.

Definition 12 $D_{a,b,c,\mu,\omega} = \{(i, j, k) \in \mathbb{Z}^3 : \mu \leq ai + bj + ck < \mu + \omega\}$ is called an *arithmetic plane* with *normal* $\mathbf{n} = (a, b, c)^T$, *approximate intercept* μ, and *arithmetic thickness* ω.

An arithmetic plane is a generalization of an arithmetic line $D_{a,b,\mu,\omega} = \{(i,j) \in \mathbb{Z}^2 : \mu \le ai + bj < \mu + \omega\}$. From Reveillés' Theorem on arithmetic lines (Reveillès 1991) we know that *naive lines* (with $\omega = \max\{|a|,|b|\}$) are the same as digital lines (which are 8-paths), and *standard lines* (with $\omega = |a| + |b|$) are the same as upper or lower digital lines (which are 4-paths, see (Rosenfeld and Klette 2001)). If $\omega = \max\{|a|,|b|,|c|\}$, then the arithmetic plane $D_{a,b,c,\mu,\omega}$ is called a *naive digital plane*; and if $\omega = |a| + |b| + |c|$, it is a *standard plane.*

The following theorem was proved in (Andres et al. 1997) by employing results from (Veelaert 1993). We provide a considerably simpler proof.

Theorem 13 *(Andres et al. 1997) Every digital plane with rational slopes is a naive plane and vice versa.*

PROOF. Consider the following identities:

$$\begin{aligned}\mu \le ai + bj + ck < \mu + \omega &\iff \frac{\mu}{|c|} \le \frac{a}{|c|}i + \frac{b}{|c|}j + k < \frac{\mu}{|c|} + \frac{\omega}{|c|} = \frac{\mu}{|c|} + 1\\ &\iff k \ge -\frac{a}{|c|}i - \frac{b}{|c|}j + \frac{\mu}{|c|} > k - 1\\ &\iff k \ge \alpha_1 i + \alpha_2 j + \beta > k - 1\end{aligned}$$

For the last identity we have used

$$\alpha_1 = -\frac{a}{|c|}, \alpha_2 = -\frac{b}{|c|}, \beta = \frac{\mu}{|c|}$$

which is equivalent to $k = \lceil \alpha_1 i + \alpha_2 j + \beta \rceil = \lfloor \alpha_1 i + \alpha_2 j + \beta + 0.5 \rfloor$. $\square$

Now assume in the definition of supporting planes that S is a set of cells in the incidence grid $\mathbb{C}_3$. We characterize upper or lower frontier planes in the grid cell model. Each 0-cell of a 3-cell c is incident with three 2-cells of c. The normals to these 2-cells form a *tripod*. There are eight different tripods. It follows that the normals of all 2-cells of any upper or lower digital frontier plane belong to one tripod.

The *main diagonal* $\mathbf{v}$ of a pair of parallel planes in $\mathbb{R}^3$ is the diagonal vector in a grid cube that has the greatest dot (inner) product with the outward pointing normal $\mathbf{n}$ of the planes (i.e., $\mathbf{v}$ has one of the eight possible directions $(\pm1, \pm1, \pm1)$ and length $\|\mathbf{v}\| = \sqrt{3}$; if there is more than one such a direction, we can choose one of them arbitrarily). The distance between both planes in main diagonal direction is called their *main diagonal distance.*

Recall (see (Rosenfeld and Klette 2001)) that in 2D, a 4-path is a 4-DSS iff its cells lie between or on a pair of supporting lines whose main diagonal distance is less than $\sqrt{2}$.

Corollary 14 *A finite 1-connected set of frontier faces of a set of 3-cells is a DPS iff all the face normals belong to one tripod, and the faces are contained between or on a pair of parallel planes whose main diagonal distance is less than $\sqrt{3}$.*

PROOF. Theorem 13 shows that a finite DPS G in the grid point model is characterized (besides connectivity) by the property that it is between two supporting planes

$$ai + bj + ck = \mu \quad \text{and} \quad ai + bj + ck = \mu + c$$

The upper supporting plane is a translation of the lower supporting plane (by translation vector $(0, 0, 1)$). The main diagonal direction of both (under the assumption $0 < a \leq b \leq c$) is $(-1, -1, +1)$, and the main diagonal distance is less than or equal to $\sqrt{3}$.

Note that points of G can lie on the lower supporting plane $ai+bj+ck = \mu$ but not on the upper one $ai + bj + ck = \mu + c$. In the corollary we also allow that points in G are on both supporting planes. It follows that the upper supporting plane is actually allowed to be translated towards the lower supporting plane (now with main diagonal distance less than $\sqrt{3}$), such that all points in G are between or on both supporting planes.

Altogether we have a necessary and sufficient characterization of DPS*s* in the grid point model based on connectivity and main diagonal distance $\sqrt{3}$.

Now consider a set of 2-cells in the grid point model. A translation by $(.5, .5, .5)$ maps all vertices of these 2-cells into grid point positions. The main diagonal distance between two parallel planes is invariant with respect to such a translation. □

Note that both parallel planes used in the above proof, are supporting planes for the given DPS in the grid cell model.

4 Height and Remainder Maps

From Theorem 13 we know that for any digital plane $I_{\alpha_1,\alpha_2,\beta}$ with rational α_1 and α_2, there exist relatively prime integers a, b, c and an integer μ such that $I_{\alpha_1,\alpha_2,\beta} = D_{a,b,c,\mu,\max\{|a|,|b|,|c|\}}$; and for any $D_{a,b,c,\mu,\max\{|a|,|b|,|c|\}}$ there exist rational slopes α_1, α_2 and an intercept β such that $D_{a,b,c,\mu,\max\{|a|,|b|,|c|\}} = I_{\alpha_1,\alpha_2,\beta}$.

Now assume $0 < a \leq b \leq c$. Without loss of generality, we consider digitizations of Euclidean planes which are incident with the origin (e.g., by assuming $\mu = 0$). If $D_{a,b,c,0,\omega}$ is a naive plane, then each voxel $(x, y, z) \in D_{a,b,c,0,c}$ projects

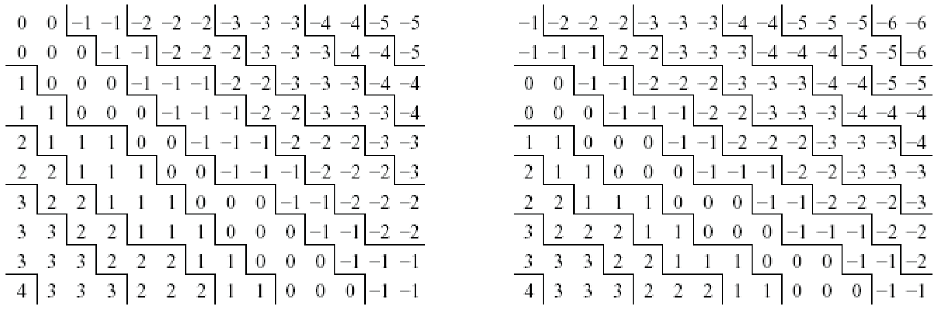


Fig. 2. (Brimkov and Barneva 2003) Two height maps, for $D_{6,7,16,0,16}$ on the left, and $D_{6,9,16,0,16}$ on the right.

onto a pixel (x, y) in the xy-plane. The condition $0 < a \leq b \leq c$ implies that there is exactly one voxel $(x, y, z) \in D_{a,b,c,0,c}$, for every $(x, y) \in \mathbb{Z}^2$. The *height map* $M^{(h)}_{a,b,c}$ is defined on $\mathbb{Z}^2$ by assigning a value z to (x, y) such that $(x, y, z) \in D_{a,b,c,0,c}$.

Figure 2 illustrates two height maps of naive planes $D_{a,b,c,0,c}$. Let $L_{a,b,c}(z_0) = \{(x, y) \in \mathbb{Z}^2 : (x, y, z_0) \in D_{a,b,c,0,c}\}$, for $z_0 \in \mathbb{Z}$. It follows that $L_{a,b,c}(z_0)$ is an arithmetic line $D(a, b, \mu, \omega)$ with $\mu = -cz_0$ and $\omega = c$; $D(a, b, \mu, \omega)$ is standard if $c = a+b$, "thicker than standard" if $c > a+b$, and "thinner than standard," but "thicker than naive" if $c < a+b$. The arithmetic lines $L_{a,b,c}(z_0)$, with $z_0 \in \mathbb{Z}$, partition $\mathbb{Z}^2$ into equivalence classes, which are all translation equivalent [1] iff a, b are relatively prime (Brimkov and Barneva 2003). See Figure 2 on the left for relatively prime integers a, b, and on the right for an example where a, b are not relatively prime.

Furthermore, $0 < a \leq b \leq c$ implies that the projections $L^{(x)}_{a,b,c}(x_0) = \{(y, z) \in \mathbb{Z}^2 : (x_0, y, z) \in D_{a,b,c,0,c}\}$ and $L^{(y)}_{a,b,c}(y_0) = \{(x, z) \in \mathbb{Z}^2 : (x, y_0, z) \in D_{a,b,c,0,c}\}$, for some $x_0, y_0 \in \mathbb{Z}$, are naive lines with approximate intercept $\mu = -ax_0$ or $\mu = -by_0$, respectively. The arithmetic lines $L^{(x)}_{a,b,c}(x_0)$, for $x_0 \in \mathbb{Z}$, partition $\mathbb{Z}^2$ into translation equivalent equivalence classes. The same holds for the arithmetic lines $L^{(y)}_{a,b,c}(y_0)$, for $y_0 \in \mathbb{Z}$; see (Debled-Renesson and Reveillès 1994) and (Debled-Rennesson 1995). [2]

Naive planes can also be represented by arrays of remainders (Debled-Rennesson 1995). Let $(x, y, z) \in D_{a,b,c,0,c}$. We assign a value $ax + by + cz$ to the grid point (x, y), i.e. the remainder modulo c. This results into a *remainder map* $M^{(r)}_{a,b,c}$. See Figure 3 for two examples. On the left we have $a = 6$ and $b = 7$, i.e. both integers are relatively prime, which results into remainders in the whole range of $0, \ldots, 15$, for $c = 16$. On the right we have $a = 6$ and $b = 9$, i.e. remainders in one equivalence class of the height map are all identical modulo $\gcd(6, 9) = 3$.

[1] $A, B \subset \mathbb{Z}^n$ are *translation equivalent* iff there is a translation vector $\mathbf{t} \in \mathbb{Z}^n$ such that $A = \mathbf{t} \oplus B$.

[2] In these works the considerations are under the restriction $gcd(a, b) = gcd(a, c) = gcd(b, c) = 1$. The general case has been handled in (Brimkov and Barneva 2003).

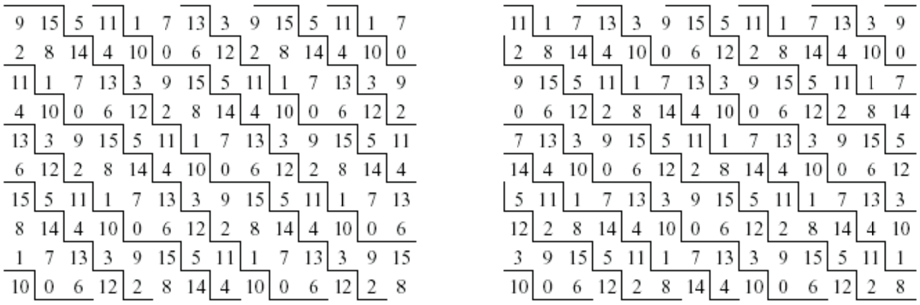


Fig. 3. Two remainder maps for the symmetric naive planes shown in Figure 2 (Brimkov and Barneva 2003).

(Brimkov and Barneva 2003) shows

Proposition 15 $M^{(r)}_{a,b,c} = M^{(r)}_{c-a,b,c} = M^{(r)}_{a,c-b,c} = M^{(r)}_{c-a,c-b,c}$, *for* $0 < a \le b \le c$.

PROOF. We prove the first equivalence, by definition of the remainder maps:

$$\begin{aligned} M^{(r)}_{c-a,b,c} &= [(c-a)x + by + cz] \pmod c \\ &= [(c-a)x \pmod c + by \pmod c + cz \pmod c] \pmod c \\ &= [ax \pmod c + by \pmod c + cz \pmod c] \pmod c \\ &= [ax + by + cz] \pmod c \\ &= M^{(r)}_{a,b,c} \end{aligned}$$

The other equivalences can be easily deduced using the same decompositions. □

This is called the *Symmetry Lemma* in (Brimkov and Barneva 2003), which defines a special type of symmetry between naive planes $D_{a,b,c,0,c}$, $D_{c-a,b,c,0,c}$, $D_{a,c-b,c,0,c}$, and $D_{c-a,c-b,c,0,c}$ (see Figure 3). If one or both parameters a and b are larger than $c/2$, then the Symmetry Lemma allows to consider w.l.o.g. symmetric naive planes $D_{a,c-b,c,0,c}$ or $D_{c-a,c-b,c,0,c}$ where the two first parameters do not exceed $c/2$. This may be useful for studying the connectivity number of a digital plane (see Section 6).

5 Periodicity and Self-Similarity

A *position* (i, j) in an array $X = (X(i, j))_{0 \le i, 0 \le j}$ is defined by a row i and a column j; $X(i, j)$ is the *element* of X at position (i, j). The elements of X are letters in an alphabet A. We continue to assume $0 \le \alpha_1, \alpha_2 \le 1$, i.e. in a quadrant step code we have $A = \{0, 1\}$.

Let $S \subseteq \mathbb{Z}_+^2 = \{(i,j) \in \mathbb{Z}^2 : i,j \geq 0\}$. The restriction $X[S]$ of X to positions in S is called a *factor of* X *on* S. If $S = \mathbb{Z}^2$ or $S = \mathbb{Z}_+^2$, we will write X instead of $X[S]$, for short.

Definition 16 A vector $\mathbf{v}$ in $\mathbb{Z}^2$ is called a *symmetry vector* for $X[S]$ iff $X(i,j) = X(\mathbf{v} + (i,j))$ for all $(i,j) \in S$ such that $\mathbf{v} + (i,j) \in S$. $\mathbf{v}$ is called a *periodicity vector* or a *period* for $X[S]$ iff for any integer k the vector $k\mathbf{v}$ is a symmetry vector for $X[S]$.

An infinite array X on $\mathbb{Z}_+^2$ is called *2D-periodic* iff there are two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{Z}^2$ such that $\mathbf{w} = i\mathbf{u} + j\mathbf{v}$ is a period for X for any $(i,j) \in \mathbb{Z}^2$ and $\mathbf{w} \in \mathbb{Z}_+^2$. X is called *1D-periodic* iff all periods of X are parallel vectors. Periodicity of a 3D set $X[S]$ where $S \subseteq \mathbb{Z}^3$ is defined analogously.

Let X be a 2D-periodic infinite array on $\mathbb{Z}_+^2$. The set of symmetry vectors of X defines (by additive closure) a sublattice Λ of $\mathbb{Z}^2$. Any basis of Λ is a *basis* of X. We say that an infinite array X on $\mathbb{Z}_+^2$ is *tiled* by a (finite) rectangular factor W if X is a pairwise disjoint repetition of W. Evidently, any 2D-periodic array on $\mathbb{Z}_+^2$ can be tiled.

It is well known that chain codes of rational digital lines are periodic while those of irrational digital lines are aperiodic (Brons 1974). About quadrant step codes we have the following result.

Theorem 17 *Any rational quadrant step code is 2D-periodic. Any irrational quadrant step code is either 1D-periodic or aperiodic.*

The formal proof of this statement is too lengthy to be included in the present survey (see (Brimkov 2002)). It particularly relies on the following well-known fact: For any rational Euclidean plane P there are (infinitely many) pairs of linearly independent "rational directions" (i.e., vectors with rational coordinates that are collinear with P). In this case the corresponding digital plane quadrant and its step code are 2D-periodic. For any irrational Euclidean plane P one of the following conditions is met. (i) P has no rational direction, i.e., there is no rational vector that is parallel to P. Note that in this case P may either contain no integer or rational points, or may contain a single point of this kind. (ii) P has a rational direction. In this case P either contains infinitely many equidistant integer points lying on a line, or P is parallel to such a line. One can show that in Case (i) the digital plane quadrant of P is aperiodic, while in Case (ii) it is 1D-periodic. The same applies to the corresponding quadrant step codes.

Any integer basis of a rational plane quadrant defines a lattice whose cells are parallelograms. Obviously the same applies to the corresponding quadrant step code. Let $ax + by + cz = d$ be a rational plane where a, b, c, d are integers and a, b, c are relatively prime.

Theorem 18 *(Brimkov 2002) The lattice cells of all bases of a rational quadrant step code have constant area* $\max\{|a|, |b|, |c|\}$.

PROOF. Without loss of generality, consider a plane $P : ax + by + cz = 0$ through the origin with $0 \le a \le b \le c$ and $c = \max\{a, b, c\}$. Let Λ be the lattice of the integer points in P. It is well-known that, given two figures in P with equal area, their orthogonal projections over the coordinate xy-plane have the same area, as well. It is also a well-known fact that all bases of Λ generate cells with equal area. Hence, it is enough to estimate the area of a parallelogram that is the orthogonal projection of a cell determined by an arbitrary basis of Λ. As a first basis vector one can chose $u = (0, c/gcd(b, c), -b/gcd(b, c))$. Then clearly a second basis vector can be $v = (gcd(b, c), y^*, z^*)$, where y^*, z^* form a solution of the linear Diophantine equation $a \cdot gcd(b, c) + by + cz = 0$, as y^* is the minimal positive integer with this property. Then the orthogonal projections of u and v over the xy-plane are respectively the vectors $u' = (0, c/gcd(b, c))$ and $v' = (gcd(b, c), y^*)$. Then the area of the corresponding cell generated by u' and v' equals $|det(u'|v')| = c$. □

Let X be an array on $\mathbb{Z}_+^2$. An $m \times n$ rectangle $S \subset \mathbb{Z}_+^2$ defines an $m \times n$-factor of X. Given two integers $k, l \ge 0$, we call a (k, l)-*suffix of* X the sub-array of X determined by its rows and columns with indexes greater than or equal to k and l, respectively. k, l-*prefix* of X is determined by the rows and columns with indexes not greater than k and l, respectively. Digital 2D ray X is called *ultimately periodic* if there are integers $k, l \ge 0$ such that the (k, l)-suffix of X has a period vector. X is *uniformly recurrent* if for every integer $n > 0$ there is an integer $N > 0$ such that every square factor of size $N \times N$ contains every square factor of size $n \times n$.

Let $P_X(m, n)$ be the number of $m \times n$-factors of X. For example, $P_X(0, 0) = 1$ for any X and $P_X(1, 1)$ is the number of distinct letters in X. We consider binary words on the alphabet $A = \{0, 1\}$. P_X generalizes the complexity function $P(w, n)$ defined (e.g.) in (Rosenfeld and Klette 2001) for 1D words w. Recall that the *complexity function* $P_w(n)$ of such a word w is defined as the number of different n-factors of w. A binary word w with $P_w(n) \le n$ for some n, is (ultimately) periodic. Sturmian words are the words that have lowest complexity among the non-ultimately periodic words, i.e., of complexity $P_w(n) = n + 1$ for any $n \ge 0$. It is also well-known that any Sturmian word is a chain code of an irrational straight line and is uniformly recurrent. In higher dimensions the situation is more complicated. For instance, it is still unknown whether a notion of minimal complexity can be reasonably defined (see (Berthé and Vuillon 2000a) and the discussion therein). To a certain extend the same applies to the notion of 2D Sturmian word. Initially it has been expected that 2D words of minimal complexity are step codes of irrational planes with no rational direction. Such words were believed to have complexity $mn + 1$. However, it has been recently shown that a 2D word of complexity $mn+1$ cannot be uniformly recurrent and does not appear to be a step code of any plane (Cassaigne 1999). Therefore, it makes sense to call 2D Sturmian words the ones that appear to be step codes of irrational planes which do not have a rational direction. Such kind of words obtained within a number of diverse digitization schemes have been investigated by S. Ito, M. Ohtsuki, L. Vuillon, V. Berthé, R. Tijdeman among others. See, e.g. (Vuillon 1998), (Arnoux et al. 2001), (Berthé and Vuillon 2000a), (Berthé and Vuillon 2000b), (Cassaigne 1999), (Berthé and Vuillon 2001) for recent contributions. Here we present some results in the framework of the plane step codes defined in Section 1.

An aperiodic irrational plane step code X still possesses certain "quasiperiodicity" and self-similarity properties. Thus every rectangular block appearing in X, appears in it infinitely many times. Moreover, all step codes of irrational planes with the same coefficients contain the same set of rectangular factors, and any rectangular factor of an irrational plane step code is also a factor of a rational plane step code. We also have that if X is an irrational plane step code, then $P_X(m,n)$ is unbounded.

An important array characteristic is its balance. Let $h(U)$ denote the number of 1's in a binary array U. Given two binary arrays U and V of the same size $m\times n$, $\delta(U,V)=|h(U)-h(V)|$ is their *balance*. A set X of arrays is said to be α-*balanced* for a certain *constant* $\alpha>0$, if $\delta(U,V)\leq\alpha$ for all pairs of $(m\times n)$-arrays $U,V\in X$, where m and n are arbitrary positive integers. An infinite array A is said to be α-*balanced* if its set of factors is α-balanced. Array balances are familiar from studies in number theory, ergodic theory, and theoretical computer science. For recent study on balance properties of multidimensional words on two or three letter alphabets see, e.g., (Berthé and Tijdeman 2002). One can show that if X is a row-wise plane step code, then $\delta(U,V)\leq m$ for any pair of $(m\times n)$-factors of X, $m,n\geq 0$ (Brimkov 2002). This bound is reachable, hence the step codes of digital planes are, overall, non-balanced.

Before presenting some other results, we provide a brief discussion on the structure of a digital plane quadrant. Recall that an (m,n)-*window* at a point $(p,q)\in\mathbb{Z}^2$ is a set of points $(i,j)\in\mathbb{Z}^2$ with $p\leq i<p+m$ and $q\leq j<q+n$. An (m,n)-*cube* at a point $(i,j)\in\mathbb{Z}^2$ of a digital plane P is the set $\{(x,y,z)\in P : i\leq x\leq i+m-1 \text{ and } j\leq y\leq j+n-1\}$. Two (m,n)-cubes at two different points (i,j) and (i',j') of a digital plane are geometrically equivalent if each of them can be obtained from the other by an appropriate translation. By $C_X(m,n)$ we denote the number of different (m,n)-cubes over the points of a digital plane X. $C_X(m,n)$ is an important parameter characterizing a digital plane structure (see, e.g., (Reveillès 1995)) and is closely related to the complexity function of a plane step code. In particular, we have that $C_X(m,n)\leq mn$. If X is rational, then $C_X(m,n)\leq lcm(q_1,q_2)$, where q_1 and q_2 are the denominators of the coefficients of x_1 and x_2 in the analytical plane representation. We always have $P_X(m,n)\geq C_X(m,n)$. If X is irrational and aperiodic, then $P_X(m,n)\geq mn$.

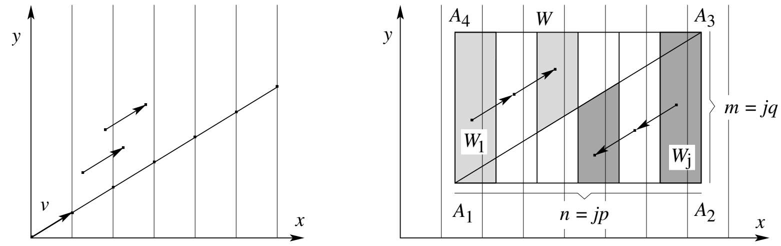


Fig. 4. Illustration to the proof of Theorem 20.

We conclude this section by listing some results related to a conjecture by M. Nivat about periodicity of infinite binary 2D words. He conjectured that if for some integers $m, n \geq 0$ an infinite bi-dimensional 0/1 array A has complexity $P_A(m, n) \leq mn$, then A has at least one period vector (Nivat 1997). Note that the converse is not true, in general: an array may be periodic but its complexity may be higher than mn (see (Berthé and Vuillon 2000a)). Only partial results for small values of m and n have been proved regarding this conjecture. In (Epifanio et al. 2003) a weaker statement is proved under the condition $P_A(m, n) \leq \frac{1}{100}mn$. For the special case of arrays that are plane step codes, we have the following results (Brimkov 2002).

Theorem 19 *A quadrant step code X has a period vector if and only if for some integers $m, n \geq 0$, $P_X(m, n) < mn$.*

If for some $m, n \geq 0$ an equality $P_X(m, n) = mn$ holds, it seems to imply the condition $P_X(m, n+1) < m(n+1)$, under which Theorem 19 applies. To prove this remains as a further task.

The next theorem provides an asymptotic result in terms of $C_R(m, n)$.

Theorem 20 *Let R be an Euclidean plane quadrant and I_R the corresponding digital plane quadrant. Then I_R has at least a 1D-period if and only if $\lim_{m,n\to\infty} \frac{C_R(m,n)}{mn} = 0$.*

PROOF. Let first $\lim_{m,n\to\infty} \frac{C_R(m,n)}{mn} = 0$. Then there exist positive integers m_0, n_0 such that for any pair of integers m, n with $m \geq m_0$ and $n \geq n_0$, we have $\frac{C_R(m,n)}{mn} < 1$, i.e., $C_R(m, n) < mn$. Then by Theorem 19, the quadrant step code i_R corresponding to R has a period vector, as I_R does.

Now let $v = (p, q, r),\ p \geq q$, be a period vector for I_R, where p, q and r are fixed integers. Let $v' = (p, q)$ be its projection on the coordinate xy-plane. Because of the symmetry of the discrete space, we can assume without loss of generality that R makes with the xy-plane an angle θ with $0 \leq \theta \leq \arctan \sqrt{2}$. Then there is a one-to-one correspondence between the voxels of I_R and the points of $\mathbb{Z}_+^2$. So to obtain quantitative estimations, one can work with projections of (m, n)-cubes over the xy-plane rather than with the (m, n)-cubes themselves. Consider the set of nonnegative integer points of the form $u^{(i)} = i \cdot v = (ip, iq)$

for $i = 0, \pm 1, \pm 2, \ldots$. They are projections on the xy-plane of points of I_R, generated by the period v. The points $u^{(i)}$ belong to a line determined by v' and induce a partition of $\mathbb{Z}_+^2$ into a set S of vertical strips delimited by the vertical rays $x = ip$, $y \geq 0$, for $i = 0, \pm 1, \pm 2, \ldots$ (Figure 4 (*Left*)). Since v is a symmetry vector of I_R, any two strips from S correspond to regions of i_R that are equivalent up to translation by vector v.

Now consider an (m, n)-window $W = A_1A_2A_3A_4$ of $\mathbb{Z}_+^2$ with $m = jp$ and $n = jq$ (see Figure 4 (*Right*)). It corresponds to an (m, n)-cube C of I_R. Partition W into j rectangles W_t $(t = 1, 2, \ldots, j)$ of width p and height jq and consider their pre-images C_t $(t = 1, 2, \ldots, j)$ from I_R under the orthogonal projection onto the xy-plane. We notice with the help of Figure 4 (*Right*) that the set of voxels from C_1 corresponding to W_1 completely determines (through translation by the vector v) all the other C_t's portions that correspond to W_t's portions over the diagonal A_1A_3. Similarly, the set of voxels from C_j corresponding to W_j completely determines (through translation by vector $(-v)$) all the other C_t's portions that correspond to W_t's portions below the diagonal A_1A_3. Thus the sets of voxels from C_1 and C_j are sufficient to completely recover the whole (m, n)-cube C. Because of the one-to-one correspondence between voxels from I_R and elements of $\mathbb{Z}_+^2$, the number of voxels in a set C_t equals the number of integer points in a strip W_t, so C_1 and C_j contain overall $2(p \cdot jq)$ voxels. From this last fact and taking advantage of the above-mentioned inequality $C_R(m, n) \leq mn$, one can easily obtain that vertical perturbations of the plane R through the window W can induce no more than $2(p \cdot jq)$ different (m, n)-cubes. Then for the ratio of $C_R(m, n)$ and mn we have the upper bound

$$\frac{C_R(m, n)}{mn} \leq \frac{2pjq}{j^2pq} = \frac{2}{j} = \frac{2p}{n}$$

which approaches 0 as n approaches infinity. □

6 Connectivity

An arithmetic line becomes 8-disconnected iff $\omega < \max\{|a|, |b|\}$. Similarly, an arithmetic plane $D_{a,b,c,\mu,\omega}$ no longer has grid points on all the vertical grid lines iff $\omega < \max\{|a|, |b|, |c|\}$.

Definition 21 Let $M \subseteq S \subseteq \mathbb{Z}^n$ $(n = 2, 3)$. M is called *α-separating* in S iff $S \setminus M$ is not α-connected $(\alpha = 4, 6, 8, 18, 26)$. Let M be α-separating in some superset of S but not β-separating in S where $\beta = 4, 6, 8, 18, 26$ and $\alpha < \beta$. Then M is said to have *β-gaps*. A set M that has no β-gaps is called *β-gapfree* and a set that has no β-gaps for any β is called *gapfree*.

The empty set is α-connected; it follows that M is not α-separating in itself.

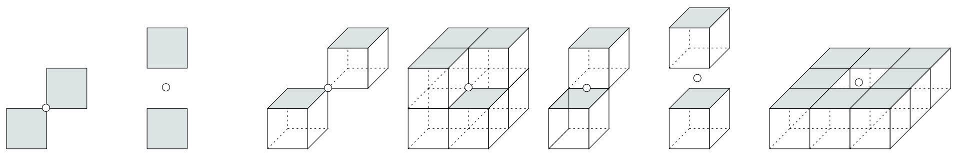


Fig. 5. From left to right: for $n = 2$ a 0-gap and a 1-gap; for $n = 3$ two 0-gaps, a 1-gap, and two 2-gaps (Brimkov et al. 2000a).

If $S = M \cup \{p\}$, then M can also not be α-separating in S. Let p, q be any two points in $\mathbb{Z}^n \setminus M$ which are not α-connected; then M is α-separating $M \cup \{p, q\}$.

In the cell model we have $\alpha, \beta \in \{0, 1, 2\}$, and M has β-gaps iff it is α-separating in some superset of S but not β-separating in S, for $\alpha > \beta$.

Figure 5 illustrates gaps. An arithmetic line is gapfree (which is equivalent to 8-gapfree) iff it is 4-connected; and it is 4-gapfree iff it is 8-connected. A naive line is 8-connected and 4-separating in $\mathbb{Z}^2$, and a standard line is 4-connected and 8-separating in $\mathbb{Z}^2$. Consider arithmetic lines $D_{a,b,\mu,\omega} = \{(i, j) \in \mathbb{Z}^2 : \mu \leq ai + bj < \mu + \omega\}$, for relatively prime integers a, b with $0 \leq a \leq b$, and integers $\omega \geq 0, \mu$. We have

(i) D is 8-disconnected iff $\omega < b$ (i.e. D has 4-gaps, see Definition 21).
(ii) D is 8-connected and has 8-gaps iff $b \leq \omega < a + b$.
(iii) D is 4-connected and gapfree iff $a + b \leq \omega$.

A standard arithmetic plane is 26-separating and gapfree; it has no 6-, 18-, or 26-gaps. A naive arithmetic plane is 6-separating but not necessarily 18- or 26-separating; it can have 18- or 26-gaps. Note that if S is not α-connected, any of its subsets is α-separating in S.

Theorem 22 *(Andres et al. 1997) Let $D_{a_1,a_2,a_3,\mu,\omega}$ be an analytical plane with $0 \leq a_1 \leq a_2 \leq a_3$ and $0 \leq \mu$. If $\omega < a_3$, the plane has 6-gaps; if $a_3 \leq \omega < a_2 + a_3$, it has 18-gaps and is 6-separating in $\mathbb{Z}^3$; if $a_2 + a_3 \leq \omega < a_1 + a_2 + a_3$, it has 26-gaps and is 18-separating in $\mathbb{Z}^3$; and if $a_1 + a_2 + a_3 \leq \omega$, it is 26-gapfree.*

PROOF. We want to show that $\omega = \sum_{i=k+1}^{n} a_i$, where $n = 3$ and $k = 2$ or 3, is the least value for which $D_{a_1,a_2,a_3,\mu,\omega}$ has no k-gaps. First we show that there is at least one k-gap for $\omega = \sum_{i=k+1}^{n} a_i - 1$. Since $gcd(a_1, a_2, a_3) = 1$, there is $y = (y_1, y_2, y_3) \in \mathbb{Z}^3$, such that $a_1y_1 + a_2y_2 + a_3y_3 = 1$. For a given digital plane $D_{a_1,a_2,a_3,\mu,\omega}$, we define its *control value* at the integer point $x = (x_1, x_2, x_3)$ as $\Pi(x, D_{a_1,a_2,a_3,\mu,\omega}) = \mu + a_1x_1 + a_2x_2 + a_3x_3$. Consider the integer point $p = (p_1, p_2, p_3)$ with $p_i = y_i$, $i = 1, 2, 3$. We have $\Pi(p, D_{a_1,a_2,a_3,\mu,\omega}) = -1$. Now consider the integer point $q = (q_1, q_2, q_3)$ with $q_i = p_i$ for $1 \leq i \leq k$ and $q_i = p_i + 1$ for $k + 1 \leq i \leq n$. By construction, p and q are k-neighbors. We have $\Pi(q, D_{a_1,a_2,a_3,\mu,\omega}) = \sum_{i=1}^{n} a_iq_i = \sum_{i=1}^{n} a_ip_i + \sum_{i=k+1}^{n} a_i = \omega$. This proves that a plane with thickness $\omega = \sum_{i=k+1}^{n} a_i - 1$ has k-gaps. Now we show that if $\omega = \sum_{i=k+1}^{n} a_i$, then $D_{a_1,a_2,a_3,\mu,\omega}$ has no k-gap. Consider two integer points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ such that $\Pi(p, D_{a_1,a_2,a_3,\mu,\omega}) = -1$ and q is a k-neighbor of p. The latter means that $q_i = p_i + e_i$, where $|e_i| \leq 1$ and $\sum_{i=1}^{n} |e_i| \leq n - k$. Then $\Pi(q, D_{a_1,a_2,a_3,\mu,\omega}) = \sum_{i=1}^{n} a_i p_i + \sum_{i=1}^{n} e_i \leq -1 + \sum_{i=k+1}^{n} a_i$, i.e., $\Pi(q, D_{a_1,a_2,a_3,\mu,\omega}) \leq \omega - 1$. Hence q cannot be on the same side of $D_{a_1,a_2,a_3,\mu,\omega}$ as p and, therefore, $D_{a_1,a_2,a_3,\mu,\omega}$ has no k-gap. $\square$

Clearly, the above proof applies also to arbitrary dimensions n.

(Reveillès 1991) stated for arithmetic lines equivalences between 8-gapfreeness and 4-connectedness, and 4-gapfreeness and 8-connectedness. This cannot be repeated for arithmetic planes. Connectivity is a translation-invariant property. W.l.o.g. we consider grid-line intersection digitizations of rational planes $ax + by + cz = 0$ which are incident with the origin, and $D_{a,b,c,\omega}$ is the corresponding arithmetic plane with thickness $\omega \in \mathbb{Z}_+$ and $a, b, c \in \mathbb{Z}_+$ with $\gcd(a, b, c) = 1$. In case of a naive plane (i.e., $\omega = \max\{a, b, c\}$) we simply write $D_{a,b,c}$.

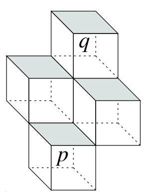
Definition 23 For $\alpha = 6, 18, 26$ and $a, b, c \in \mathbb{Z}_+$, let

$$\Omega_\alpha(a, b, c) = \max\{\omega : \ D_{a,b,c,\omega} \text{ is } \alpha - \text{disconnected}\}$$

be the *α-connectivity number* of the class of all arithmetic planes $D_{a,b,c,\omega}$, with $\omega \in \mathbb{Z}_+$.

In other words, $\omega = \Omega_\alpha(a, b, c) + 1$ is the smallest integer such that $D_{a,b,c,\omega}$ is α-connected. Evidently, $\Omega_\alpha(a, b, c) \leq \Omega_\beta(a, b, c)$ if $\alpha \geq \beta$, with $\alpha, \beta \in \{4, 18, 26\}$. Naive planes are always 26-connected, i.e. $\Omega_{26}(a, b, c) \leq \max\{a, b, c\}$, and standard planes are always 6-connected, i.e. $\Omega_6(a, b, c) \leq a + b + c$. Connectivity numbers remain constant when permuting a, b, c, e.g., $\Omega_\alpha(a, b, c) = \Omega_\alpha(b, c, a)$.

Assume the grid cell model. A pair of voxels $p = (i, j, k)$ and $q = (i + 1, j + 1, k + 2)$ (see Figure 6 (*Left*)) defines a *jump*. A naive plane $D_{a,b,c,\mu,c}$ (with $c = \max\{a, b, c\}$) contains a jump iff $c < a + b$ (Brimkov and Barneva 2002).



−1	−2	−2	−3	−3	−3	−4	−4	−5	−5	−6	−6	−7	−7
−1	−1	−1	−2	−2	−3	−3	−4	−4	−5	−5	−6	−6	−6
0	0	−1	−1	−2	−2	−3	−3	−4	−4	−4	−5	−5	−6
1	0	0	−1	−1	−2	−2	−2	−3	−3	−4	−4	−5	−5
1	1	0	0	0	−1	−1	−2	−2	−3	−3	−4	−4	−5
2	2	1	1	0	0	−1	−1	−2	−2	−3	−3	−3	−4
3	2	2	1	1	0	0	−1	−1	−1	−2	−2	−3	−3
3	3	2	2	1	1	1	0	0	−1	−1	−2	−2	−3
4	3	3	3	2	2	1	1	0	0	−1	−1	−2	−2
5	4	4	3	3	2	2	1	1	0	0	0	−1	−1

Fig. 6. *Left:* A jump; *Right:* Height map of the naive plane $D_{5,7,11,0,11}$: the 8-connected set of pixels (shown in gray) is a projection of a 26-disconnected set of voxels of this naive plane (Brimkov and Barneva 2003).

Figure 6 (*Right*) illustrates a naive plane where 8-connected sets of pixels in a height map may be projections of 26-disconnected sets of voxels in the naive plane. The Symmetry Lemma (Proposition 15) allows to transform such naive planes into symmetric (in the sense of the Symmetry Lemma) naive planes where $c < a + b$ is not true anymore, which also allows to conclude:

Proposition 24 $\Omega_{26}(a,b,c) = \Omega_{26}(c-a,b,c) = \Omega_{26}(a,c-b,c) = \Omega_{26}(c-a,c-b,c)$*, for relatively prime integers* a, b, c *with* $0 < a \leq b \leq c$*.*

The rest of this section reviews some results from (Brimkov and Barneva 2003). The following theorem provides reachable upper and lower bounds for the connectivity number.

Theorem 25 $a - 1 \leq \Omega_{26}(a,b,c) \leq b - 1$*, if* $a + b < c < a + 2b$*.*

(Brimkov and Barneva 2003) also provides an algorithm computing $\Omega_{26}(a,b,c)$ with $O(a \log b)$ arithmetic operations, where $0 \leq a \leq b \leq c$. Within a model with a unit cost floor operation, the algorithm complexity is $O(a)$.

Theorem 26 *(Brimkov and Barneva 2003) Let* a, b, c *be relatively prime integers with* $c \geq a + 2b$ *and* $a > 0$*. Then* $\Omega_{26}(a,b,c) = c - a - b + \gcd(a,b) - 1$*.*

PROOF. Let A be a 2D array (finite or infinite) and $p = (x_0, y_0)$, $q = (x_m, y_m)$ two points of A. Let, for definiteness, $x_0 \leq x_m$ and $y_0 \leq y_m$. The sequence of points $P = \langle (x_0, y_0) = p, (x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m) = q \rangle$ is a *stairwise path* between p and q if the coordinates of two consecutive points (x_i, y_i) and (x_{i+1}, y_{i+1}), $0 \leq i \leq m-1$, satisfy either $x_{i+1} = x_i, y_{i+1} = y_i + 1$, or $x_{i+1} = x_i + 1, y_{i+1} = y_i$. The number m is the *length* of the path. For all other possible mutual locations of p and q, a stairwise path is defined similarly (see Figure 7 (*Left*)).

Consider now the remainder map $M^{(r)}_{a,b,c}$ together with its equivalence classes described above. The points of $M^{(r)}_{a,b,c}$ which contain the value $\Omega_{26}(a,b,c)$ are called the *plugs* of $M^{(r)}_{a,b,c}$. The points containing the maximal possible value $c-1$ are the *maximal points* of $M^{(r)}_{a,b,c}$. Assume for a moment that c is "enough large" compared to a and b. More precisely, suppose that $c \geq a + 2b$. Then the discrete lines corresponding to the equivalence classes are thicker than standard. In particular, if $c = a+2b = (a+b)+b$, then a particular equivalence class C is a disjoint union of one standard and one naive line. Note that in this case there are two different possible partitions of this kind: one can consider the standard line to be above the naive, and vice versa. In the first case we call the standard line *upper standard line* for the class C, while in the second case we call it *lower standard line* for C. Similarly, if $c > a+2b$, then C can be partitioned in two different fashions into disjoint union of one standard line and another line which is thicker than naive. Consider then a class C which contains maximal points of $M^{(r)}_{a,b,c}$, where $c \geq a + 2b$. We have $C = S \cup L$, where S is the standard line containing maximal points of $M^{(r)}_{a,b,c}$, and L is a discrete line that is naive or thicker than naive. A point $P \in S$ with a minimal value is called a *core of the class* C (see Figure 7 (*Right*)). Keeping in mind the properties of $M^{(r)}_{a,b,c}$, we can state the following lemma.

Lemma 27 *Let P_1 and P_2 be two consecutive maximal points belonging to an equivalence class C. Let $S \subseteq C$ be the standard line containing P_1 and P_2, and $\bar{S}(P_1, P_2) \subset S$ the stairwise path between P_1 and P_2. Then:*

(1) All points of S have different values;

(2) $\bar{S}$ contains $\frac{a+b}{\gcd(a,b)}$ points with values $c-1, c-1-\gcd(a,b), c-1-2\gcd(a,b), \dots, f$, where the last value f is equal to

$$f = c - 1 - \left(\frac{a+b}{gcd(a,b)} - 1\right)\gcd(a,b) = c - a - b + \gcd(a,b) - 1$$

See Figure 7 (*Right*). To complete the proof of the theorem, let the points $P_1, P_2 \in C$, the standard line S, and the stairwise path $\bar{S}(P_1, P_2)$ be as in Lemma 27. This last lemma implies that $\bar{S}$ contains a unique core of C. Clearly, when ω decreases starting from $c-1$ and going downwards, first the points from the standard line S will vanish from $M^{(r)}_{a,b,c}$. Consider first what happens when $c = a + 2b$. As already discussed above, the complement of S to C is a naive line L which is "below" S. Moreover, the mutual location of S and L within the class C implies the following property: The 4-neighbors of any pixel from S are points which belong either to S or to L. See Figure 7 (*Right*).

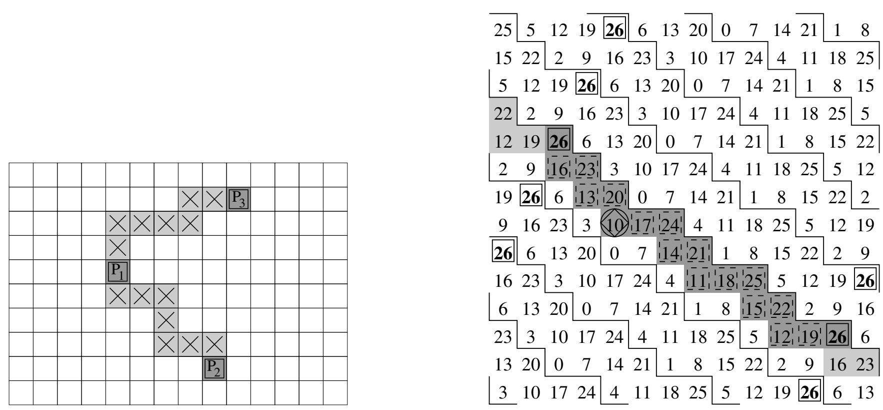


Fig. 7. *Left:* Two stairwise paths marked by shadowed $\times$ sign: one between the points P_1 and P_2, and another between the points P_1 and P_3; *Right:* A stairwise path between two maximal points of value 26 in array $A(7, 10, 27)$. The path (in dark gray) is a part of an upper standard line (in gray) through the two maximal points. The core of the class has value 10. It coincides with a plug of $A(7, 10, 27)$. A core is marked by $\bigcirc$ and a plug by $\diamond$.

Therefore, if the points of S are removed from C, all points of the naive line L will be disconnected from the points of the next equivalence class "above" C. Obviously, this will also hold when $c > a + 2b$. All equivalence classes are discrete lines and therefore are periodic. The period length of a class is equal to $a + b$ which is the length of the path between two consecutive maximal points of C. Therefore, the disconnectedness considered above propagates along all the class C. On the other hand, the array of remainders $M^{(r)}_{a,b,c}$ is periodic, as the class C appears periodically in a way that if we start counting from it, every $gcd(a, b)$th class is equivalent to C. Thus we obtain that if $c \geq a + 2b$, the array $M^{(r)}_{a,b,c}$ becomes disconnected when the points of the standard line S are removed from it.

What remains to show is that $\Omega_{26}(a, b, c) = c - a - b + gcd(a, b) - 1$. Clearly, the value of $\Omega_{26}(a, b, c)$ is equal to the value of a core of a class C that contains maximal values. In other words, we have that the set of plugs of $M^{(r)}_{a,b,c}$ coincides with the set of the cores of all classes containing maximal elements. If $gcd(a, b) = 1$, then $\Omega_{26}(a, b, c) = c - a - b = c - a - b + gcd(a, b) - 1$, since $M^{(r)}_{a,b,c}$ becomes disconnected when points with values $c-1, c-2, \ldots, c-a-b$ are removed from it. Now let $gcd(a, b) = d \neq 1$. Consider again the points in a stairwise path $\bar{S}(P_1, P_2)$ between two consecutive maximal points in a class C. Then part 2 of Lemma 27 implies that if $c \geq a + 2b$, then $\Omega_{26}(a, b, c) = c - a - b + gcd(a, b) - 1$. □

This theorem combined with Proposition 24 allows to derive further explicit solutions, such as

$$\Omega_{26}(a, b, c) = b - a + \gcd(a, c - b) - 1\,, \text{ if } c < 2b - a$$
$$\Omega_{26}(a, b, c) = b + a - c + \gcd(c - b, c - a) - 1\,, \text{ if } c < a + b/2$$

and the lower bound

$$\Omega_{26}(a, b, c) \geq c - a - b + \gcd(a, b) - 1 \text{ for any and } a, b, c$$

7 Algorithms

Theoretical research on digital planarity is naturally driven by important practical applications in image analysis, pattern recognition and volume modeling. In this section we review some basic algorithms for digital plane recognition, digital surface segmentation, and digital polyhedra generation.

Let S be a digital plane segment defined by an Euclidean plane $\Gamma(\alpha_1, \alpha_2, \beta)$ with $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 1$ and $0 \leq \beta < 1$.

Definition 28 The preimage of a DPS S is the set of points $(\alpha_1, \alpha_2, \beta) \in [0,1]^2 \times [0,1[$, such that $S \subset I_{\alpha_1,\alpha_2,\beta}$.

In other words, the preimage is the set of Euclidean planes whose digitizations contain S. According to this definition and the discussion related to Theorem 9, the preimage is the solution of a system of linear inequalities with unknowns α_1, α_2, and β. Thus it is a convex polyhedron (possibly empty).

In dimension 2, the analysis of the preimage structure allowed to design efficient recognition algorithms (Dorst and Smeulders 1984) (McIlroy 1985) (Lindenbaum and Bruckstein 1993). Indeed, the preimage associated to a given 8-arc has a robust arithmetic structure (describable by means of Farey cells). Moreover, the number of vertices of this domain is bounded by 4. Beside the arithmetic properties of the preimage, the bound on the number of vertices induces a linear-time (i.e., the processing time for each new vertex is a constant) on-line algorithm to compute and update the 2D preimage, and thus to decide whether an $8-$arc is a digital straight segment or not.

In 3D, few works have studied the arithmetic properties of the preimage and its geometry. Consider the digital plane segment $S \subset D_{a,b,c,\mu,c}$ (again, without loss of generality, we suppose that $0 \leq a \leq b < c$). From the remainder map associated to S (see Section 4), we can define the *lower* (resp. *upper*) *leaning points* whose remainder $r = ax + by + cz$ is μ (resp. $\mu + c - 1$). For the sake of clarity, we suppose that S contains at least three upper and three lower leaning points. The analysis from (Coeurjolly et al. 2003) is based on the following proposition.

Proposition 29 *(Coeurjolly et al. 2003) Let $S \subset D_{a,b,c,\mu,c}$ be a piece of a naive plane. Then, the preimage of S, denoted $\mathcal{P}(S)$, containing all the Euclidean planes in the parameter space has the following properties :*

- *The points $v_\mathrm{l} = (\frac{a}{c}, \frac{b}{c}, \frac{\mu}{c})$ and $v_\mathrm{u} = (\frac{a}{c}, \frac{b}{c}, \frac{\mu+1}{c})$ are vertices of $\mathcal{P}(S)$. They correspond to the lower and the upper supporting planes $ax + by + cz = \mu$ and $ax + by + cz = \mu + c$ in the primal space;*
- *The preimage faces adjacent to v_l (resp. v_u) result from the vertices of the 2D convex hull of lower (resp. upper) leaning points in S.*

An illustration to the 2D convex hulls is given in Figure 8. As a consequence of this proposition, the number of preimage faces is at least the number of vertices of the convex hull of the upper 2D leaning points plus the number of vertices of the convex hull of the lower 2D leaning points. In (Coeurjolly et al. 2003), the authors also prove that for a given class of digital plane segments, the preimage does not have other faces than those induced by leaning points. However, a general result with a specific recognition algorithm is still a challenging work.

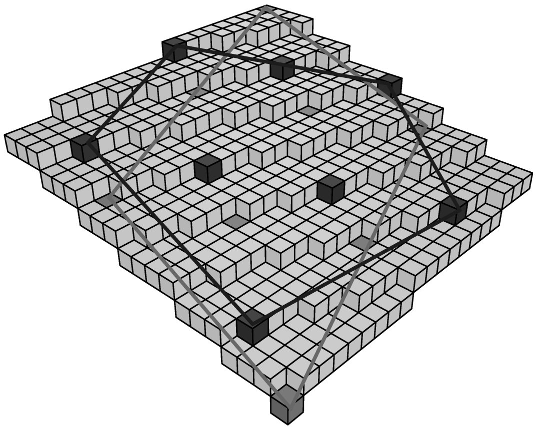


Fig. 8. Illustration of a subset of a digital plane $D_{7,17,57,0,57}$ with its lower and upper convex hulls on the supporting planes.

7.2 DPS Recognition and Digital Surface Segmentation

Theorem 11 has been used in (Stojmenović and Tosić 1991) to suggest a DPS recognition algorithm based on convex hull separability. The recognition of DPS*s* in grid adjacency models, i.e. considered to be subsets of $\mathbb{Z}^3$, is also discussed in (Veelaert 1994) (applying the characterization by evenness as given above), (Klette et al. 1996) (recognition by least-square optimization), and (Megiddo 1984)(Preparata and Shamos 1985)(Vittone and Chassery 2000)(Buzer 2002) (linear programming when the dimension is fixed). (Debled-Renesson and Reveillès 1994) proposes an approach based on tests for existence of lower and upper supporting planes (called lower and upper *oblique planes*) for the given set of points.

Table 1 sums up different algorithms and their computational costs. All complexity bounds are given with respect to the number n of grid points in S. The fourth column indicates if the algorithm perform a coplanarity test ($\mathcal{T}$) or may lead to a complete recognition ($\mathcal{R}$). The references are sorted chronologically.

(Françon et al. 1996) suggests a recognition method for DPS*s* by converting the problem to a system of n^2 linear inequalities, where n is the cardinality of the given set of points. The system is solved by the Fourier elimination algorithm. One can also apply CDD algorithm for solving systems of linear inequalities by successive intersection of half-spaces defined by inequalities (Fukuda and Prodon 1996). A very efficient incremental algorithm based on a similar approach is proposed in (Klette and Sun 2001). Typical timing results for these three versions are shown in Figure 9, using a polyhedrized digital ellipsoid at grid resolutions ranging from 10 to 100. In what follows we present

Main reference	Description	Complexity	$\mathcal{T}$ or $\mathcal{R}$	Comments
(Kim 1984)	Detection of a support plane	$O(n^4)$	$\mathcal{T}$	based on an incorrect theorem
(Megiddo 1984)	Linear programming	$O(n)$	$\mathcal{T}$	
(Preparata and Shamos 1985)	Linear programming	$O(n \log n)$	$\mathcal{R}$	
(Kim 1991)	Detection of a support plane	$O(n^2 \log n)$	$\mathcal{T}$	optimization of (Kim 1984), also based on an incorrect theorem
(Stojmenović and Tosić 1991)	Convex hull separability	$O(n \log n)$	$\mathcal{T}$	
(Veelaert 1994)	Evenness property	$O(n^2)$	$\mathcal{T}$	rectangular DP
(Debled-Renesson and Reveillès 1994)	Arithmetic structure	n.a.	$\mathcal{R}$	rectangular DP
(Reveillès 1995)	Arithmetic geometry	$O(n)$	$\mathcal{R}$	rectangular DP
(Vittone and Chassery 2000)	Linear programming and Farey series	$O(n^3 \log n)$	$\mathcal{R}$	preimage computation with arithmetic solutions
(Buzer 2002)	Linear programming for DPS recognition	$O(n)$	$\mathcal{T}$	on-line algorithm

Table 1
Algorithms for DPS recognition.

more in detail the algorithm from (Klette and Sun 2001), which appears to be superior to the others.

Algorithm KS2001

Π is called a *supporting plane* of a finite set of faces if the faces are all in one of the closed halfspaces defined by Π and their diagonal distances to Π are all less than $\sqrt{3}$. If the set of faces has $n \geq 4$ vertices, Π must be incident with three non-collinear vertices and all the other vertices must lie on or on one side of Π. A set of faces can have more than one supporting plane.

In other words, a DPS in the incidence grid can be assumed to be a 1-connected

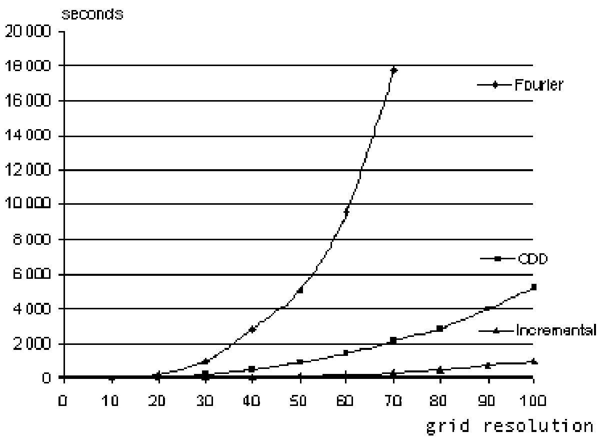


Fig. 9. Running times of three DPS recognition algorithms on a PIII 450 running Linux. (*L. Papier* provided the Fourier elimination program.)

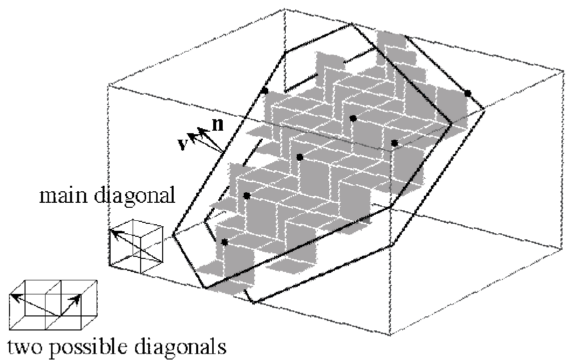


Fig. 10. A DPS; the main diagonal distance between the two parallel planes is less than $\sqrt{3}$.

set of 2-cells in the frontier of a 6-region of voxels; considered together with its incident 0- and 1-cells, it is a 2D Euclidean cell complex. A *simply-connected DPS* consists of faces whose union is homeomorphic to the unit disk, i.e. it is a 1-simply-connected set of 2-cells. Figure 10 shows a DPS; $\mathbf{n}$ is its normal and $\mathbf{v}$ is the vector in the main diagonal direction.

If we are given the frontier of the projection of the DPS onto one of the two parallel planes, it is possible to reconstruct the DPS in 3D space (up to a translation in the normal direction to the planes).

Let $\mathbf{v}$ be the vector of length $\sqrt{3}$ in main diagonal direction and let $\mathbf{n}$ be an outward pointing normal to the pair of parallel planes. Furthermore, for grid vertex $\mathbf{p}$ incident with the DPS, let $\mathbf{v} \cdot \mathbf{p} = d_{\mathbf{p}}$ be the equation of a plane with normal $\mathbf{v}$ and incident with $\mathbf{p}$. The vertices $\mathbf{p}$ of the grid faces of a DPS must satisfy

$$0 \leq \mathbf{n} \cdot \mathbf{p} - d_{\mathbf{p}} < \mathbf{n} \cdot \mathbf{v} \tag{1}$$

Let $\mathbf{n} = (a, b, c)$. The scalars a, b, c may have different signs, but since $\mathbf{n}$ and $\mathbf{v}$ must point in the same direction "modulo a directed diagonal," we can assume w.l.o.g. that $a, b, c > 0$. Equation (1) then becomes

$$0 \le ax + by + cz - d_{\mathbf{p}} < a + b + c \tag{2}$$

Hence, a DPS in the grid-cell model is equivalent (by mapping vertices into grid points) to a finite 6-connected set of grid points in a standard digital plane (see Definition 12), with $\nu = d_{\mathbf{p}}$ and $\omega = a + b + c$.

In addition to checking the tripod condition (which is easy), the task of DPS recognition (in the grid cell model) can be solved by answering the following question: Given n vertices $\{p_1, p_2, \ldots, p_n\}$, does each p_i with $d_i = \mathbf{v} \cdot p_i$ satisfy Equation (1), i.e. do we have

$$0 \le \mathbf{n} \cdot p_i - d_i < \mathbf{n} \cdot \mathbf{v} \quad \text{for } i = 1, \ldots, n\,? \tag{3}$$

The incremental algorithm repeatedly updates a list of supporting planes; if the list is empty, the set of points is not a DPS. The updating step is as follows: If we have $n \ge 0$ points, we add an $(n+1)$st point iff the list of supporting planes remains non-empty. To test this, we first check the new point against each of the listed supporting planes to see if it is on the same side of the plane as the other points and within the allowed diagonal distance. If these conditions are not satisfied, we delete the plane from the list. We then construct new supporting planes by combining the new point with pairs of existing points. A new supporting plane is added to the list if all $n + 1$ points satisfy the conditions. The set of points is accepted as a DPS iff the final list of planes is non-empty. The updating step is time-efficient because we can restrict the tests to points that have extreme positions in any of the eight diagonal directions.

A given surface S consists of edge-connected faces. These faces can be represented by a face graph whose nodes are the faces and where each node has

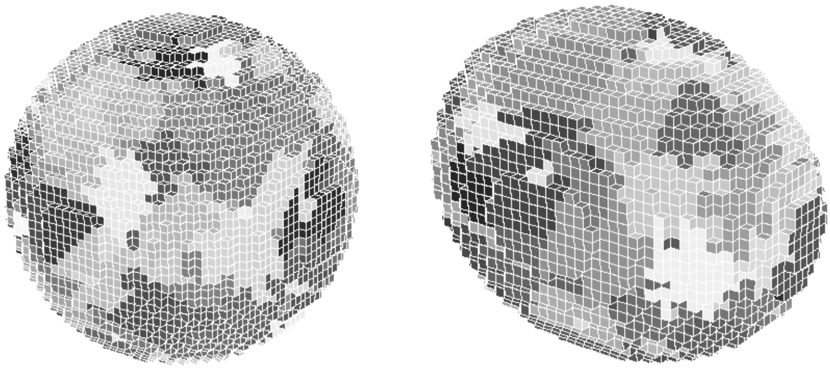


Fig. 11. Agglomeration into DPSs of the faces of a sphere and an ellipsoid (grid resolution $h = 40$).

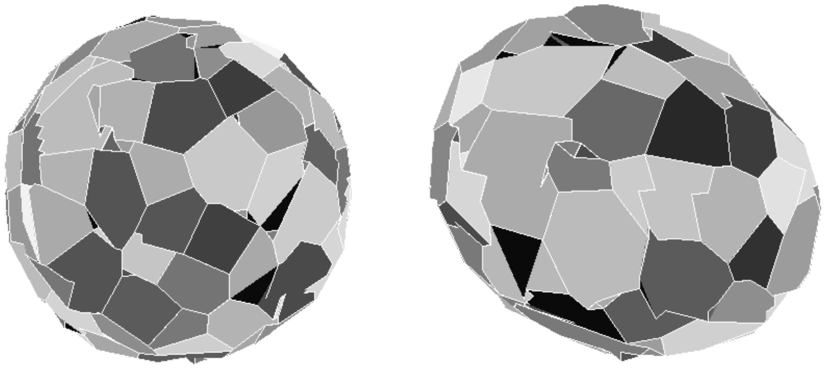


Fig. 12. A polyhedrized sphere and ellipsoid.

four pointers to its edge-adjacent faces. The face graph can be constructed using (e.g.) the Artzy-Herman surface tracing algorithm.

We can perform a breadth-first search of the face graph to agglomerate the faces into DPSs. The second process is implemented using two queues. The first is called a *seeds queue*; it contains all the faces found by the search which do not belong to any yet recognized DPS.

A face is inserted into the seeds queue if it cannot be added to the current DPS. The next DPS starts from a face chosen from the seeds queue; the choice of this face determines how the DPS "grows." The second queue is used to maintain the breadth-first search. "Growing a DPS" looks like propagating a "circular wave" on S from a center at the original seed face.

We try to add an adjacent face to the current DPS by testing each vertex of the face that is not yet on the DPS. If all four vertices pass the test, the face is added to the DPS and deleted from the seeds queue (if it was on that queue). Otherwise, we insert the face into the seeds queue and try another adjacent face. If no more adjacent faces can be added, we start a new DPS from a face on the seeds queue.

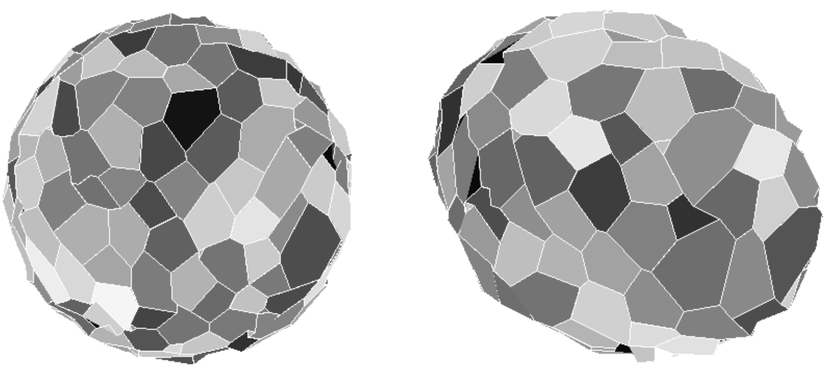


Fig. 13. The polyhedrized sphere and ellipsoid where the breadth-first search depth is restricted to 7.

A list of the frontier vertices of each DPS is maintained during the agglomeration process, not only to simplify the tests of whether a new vertex can be added, but also to maintain the topological equivalence of the DPS to a unit disk. This ensures that the frontier always remains a simple polygon, so that the algorithm constructs only simply-connected DPSs. (This condition can be removed, if desired.)

Figure 11 illustrates results of the agglomeration process for a digitized sphere and for an ellipsoid with semi-axes 20, 16, and 12. Faces that have the same gray level belong to the same DPS. The numbers of faces of the digital surfaces of the sphere and ellipsoid are 7,584 and 4,744 respectively. The numbers of DPSs are 285 and 197; the average sizes of these DPSs are 27 and 24 (faces).

To complete the polyhedrization process, we set all the face vertices that are incident with at least three of the DPSs to be vertices of the polyhedron. Figure 12 shows the final polyhedra for the sphere and ellipsoid. Note that these polyhedra are not simple; their surfaces are not hole-free.

Restricting the depth of the breadth-first search changes the polyhedrization from global to local and results in "more uniform" polyhedra. Figure 13 shows results when the depth is restricted to 7. The number of small DPSs is reduced and the sizes of the DPSs are more evenly distributed. The respective numbers of DPSs are 282 and 180 and their average sizes are 27 and 26; note that these are nearly the same as in the unrestricted case.

As mentioned above, the output of Algorithm KS2001 may not be a valid polyhedron from a topological point of view. Therefore, the next action to perform is making-up this polyhedron while preserving the reversibility (up to a digitization process) of all elements – faces, edges, and vertices. Below we sketch an algorithm from (Coeurjolly et al. 2004) that addresses the problem of a topologically correct and reversible polyhedrization.

Algorithm CGS2004

The main idea is to simplify the polyhedron obtained by a Marching-Cubes (MC) algorithm (Lorensen and Cline 1987), using information about the digital surface segmentation. With a reference to (Lachaud and Montanvert 2000), the triangulated surface obtained by the MC algorithm is a combinatorial 2-manifold. In other words, the surface is closed, hole-free and without self-crossing. Furthermore, the object boundary quantization (OBQ) of this polyhedron is exactly the input binary object.

Let us consider a voxel p from the object boundary and a voxel q from the background, such that the L_1 distance between p and q is 1. Both voxels de-

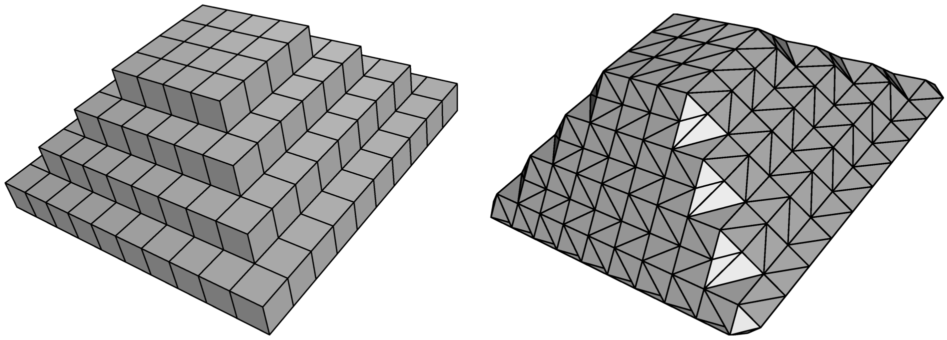


Fig. 14. A $\{0,1\}$-binary object and a Marching-Cubes surface obtained with an iso-level in $]0,1[$.

fine a segment $]pq[$ (see Figure 15-*left*). Note that every MC vertex belongs to a distinct $]pq[$ segment [3]. More precisely, a MC vertex can be attached to each boundary surface element. In (Lachaud and Montanvert 2000) it is proved that the MC surface is a combinatorial 2-manifold, independently of the position of the vertices in the $]pq[$ segments. Furthermore, a vertex displacement along the $]pq[$ does not change the reversibility property.

To link all these properties to the polyhedrization problem, we consider a set S of voxels from the object boundary such that S is a DPS, and π is a Euclidean plane from the DPS preimage (we also suppose that π does not belong to the preimage boundary). It can be proved that π crosses each segment $]pq[$ for each p in S. Let $\mathcal{P}$ be the polyhedron given by projecting the MC vertices associated to S onto π along the $]pq[$ segments. Then it can be proved that $\mathcal{P}$ is a combinatorial 2-manifold that still has the reversibility property. Moreover, all the triangles associated to the set S are coplanar. Finally, the next step of the algorithm consists of merging the coplanar triangles associated to S while preserving the topological property of the surface. We repeat the projection operation and merging steps for every recognized digital plane.

The output of the algorithm is a polyhedron such that a large facet is associated to each recognized DPS. The facets of the polyhedron are stitched together by strips of triangles. Those triangles are called *non-homogeneous* in (Coeurjolly et al. 2004) because their three vertices do not belong to the same digital plane. Finally, the polyhedron is a combinatorial 2-manifold and possesses the reversibility property.

7.3 Digital Polyhedra Generation

In this section we briefly consider certain problems that are in a sense reverse to those of the previous section. One of these is DPS generation. Usually straightforward methods for its solution directly follow from the particular

[3] $]pq[$ denotes an open segment, *i.e.* the extremities are excluded.

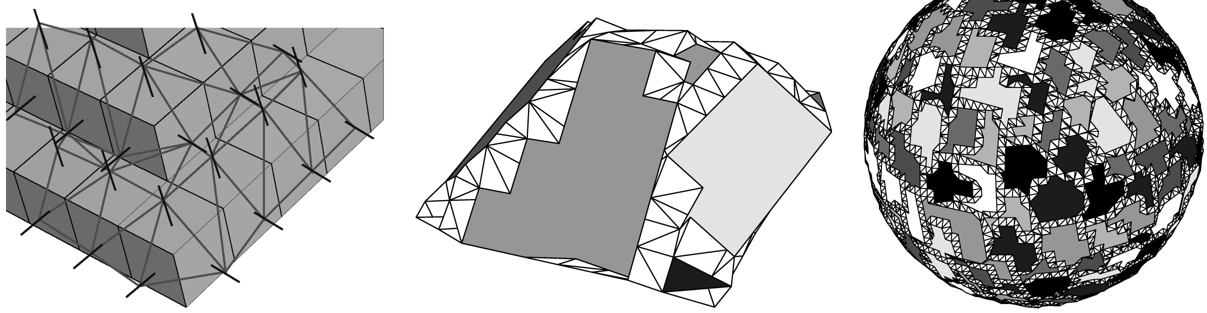


Fig. 15. *From left to right:* links between MC vertices and $]pq[$ segments, final result on the object of Figure 14, and result on a sphere of radius 25.

definition of a digital plane. See, e.g., (Debled-Renesson and Reveillès 1994) for an algorithm based on Reveillès definition of arithmetic planes. A related problem is the digitization (scan-conversion) of a given space polygon. An efficient practical algorithm has been proposed in (Kaufman 1987). Algorithms involving "supercovers" (i.e., "thick" digitizations including all voxels intersected by the given polygon) have been proposed in (Andres et al. 1997a). Discrete linear manifolds within a "standard model" (i.e., based on standard planes) have been defined in (Andres 2003).

For various applications in surface modeling it is reasonable to work with an appropriate polyhedral approximation of a given surface rather than with the surface itself. Often this is the only possibility since the surface may not be available in an explicit form. Thus having suitable algorithms for digitizing a polyhedral surface is of significant practical importance. The abovementioned supercover approach has been applied to polyhedra digitization (Andres et al. 1997b). The faces of the obtained digital polyhedra admit analytical description. They are portions of planes' supercovers that are thicker than the (naive) digital planes. As discussed in the literature, the optimal ground for polyhedra digitization is naturally provided by the naive digital planes. However, it has been unclear for a long time how to define a "naive" digital polygon and especially its edges, so that the overall discretization to admit no gaps along the edges of the resulting digital polyhedron. These theoretical obstacles have been recently overcome by employing relevant mathematic approaches. Specifically, three different algorithms have been proposed. The first one (Barneva et al. 2000) is based on reducing the 3D problem to a 2D one by projecting the surface polygons on suitable coordinate planes, next digitizing the obtained 2D polygons, and then restoring the 3D discrete polygons. The generated discrete polygons are portions of the naive planes associated with the facets of the surface. Another algorithm (Brimkov and Barneva 2002) is based on introducing new classes of 3D lines and planes (called *graceful*) which are used to approximate the surface polygons and their edges, respectively. The algorithm from (Brimkov et al. 2000b) approximates directly every space polygon by a discrete one, which is again the thinnest possible, while the polygons' edges are approximated by the thinnest possible naive 3D straight lines defined algorithmically in (Kaufman and Shimony 1986) and analytically in (Figueiredo and Reveillès 1995) and (Brimkov et al. 2000b). All these algorithms assure 6-gapfree discretizations. They run in time that is linear in the number of the generated voxels, which are stored in a 2D array. Moreover, the generated 3D discrete polygons admit analytical description.

In the remainder of this section we briefly describe the algorithm from (Brimkov et al. 2000b). Our choice is dictated by the fact that this algorithm provides an "optimal solution" while being optimally fast and using memory space of optimal order. In fact, the obtained discretization appears to be *minimally thin*, in a sense that removing an arbitrary voxels from the discrete surface leads to occurrence of a 6-gap in it.

Algorithm BBN2000

For the sake of simplicity, consider a polyhedral surface which is a mesh of triangles. As mentioned, the triangles' sides are modeled by naive 3D lines and their interiors by naive planes. Naive 3D lines have been first defined algorithmically in (Kaufman and Shimony 1986). Given a Euclidean straight line L determined by the vector (a, b, c) with $0 \leq a \leq b \leq c$, the digitization of L by truncation is the set of voxels (x, y, z) with coordinates $x = \lfloor \frac{ai}{c} \rfloor, y = \lfloor \frac{bi}{c} \rfloor, z = i,\ i \in \mathbb{Z}$. This digital line is 26-connected and "minimal" in a sense that the removal of any element splits the set into two separate 26-connected components. It can analytically be defined by $0 \leq -cx + az + \lfloor \frac{c}{2} \rfloor < c, 0 \leq -cy + bz + \lfloor \frac{c}{2} \rfloor < c$. Such a digital 3D line is called *regular* and denoted by L_R. It is centered about the continuous line L and every voxel of L_R is intersected by L. A regular naive line through two points A and B is denoted $L_R(AB)$. See Figure 16 (*Left*).

The construction of the triangle interior is somewhat more sophisticated. Recall that an arithmetic plane $P = P_{a,b,c,\mu,\omega}$ is *functional* over a coordinate plane, say, xy, if for any pixel (x, y) from xy there is exactly one voxel belonging to P. The coordinate plane xy is called *functional plane* for P and denoted by π_P. Consider first a 2D Euclidean triangle $\triangle A'B'C'$ in the xy-plane. We define the *integer set* $I_{2D}\triangle A'B'C'$ of $\triangle A'B'C'$ as the set of all integer points which belong to the interior *or* the sides of $\triangle A'B'C'$. Thus, in particular, the vertices A', B', and C' belong to $I_{2D}\triangle A'B'C'$ (see Figure 16 (*Middle*). The 3D triangle is a portion of a special kind of naive plane $P_{a,b,c,\mu+[\frac{c}{2}],c)}$, centered about the Euclidean plane and called *regular*. A regular plane through the points A, B, C is denoted P_R^{ABC}. Then an *integer set* of a 3D triangle $\triangle ABC$ is defined as follows. Let A', B', and C' be the projections of A, B and C onto $\pi_{P_R^{ABC}}$ and $I_{2D}\triangle A'B'C'$ the integer set of $\triangle A'B'C'$. Then the *integer set* $I_{3D}\triangle ABC$ of $\triangle ABC$ is the set of voxels belonging to P_R^{ABC} and whose projections on $\pi_{P_R^{ABC}}$ constitute exactly the set $I_{2D}\triangle A'B'C'$. Note that the centers of the voxels of the integer set of $\triangle ABC$ do not necessarily belong

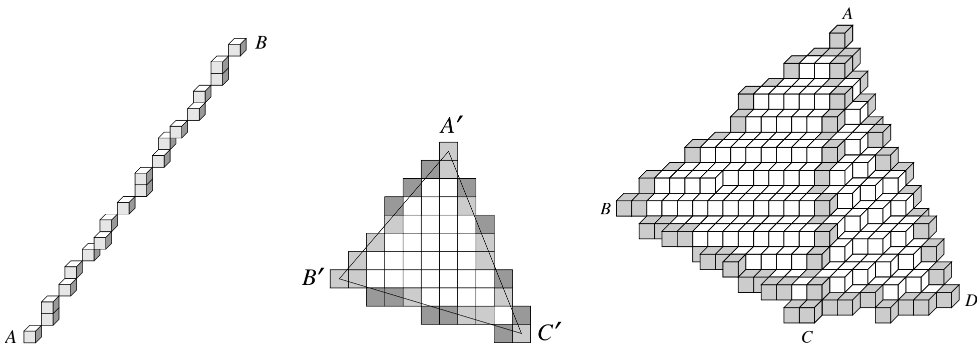


Fig. 16. *Left:* A regular naive 3D line between the points $A = (0, 0, 0)$ and $B = (11, 13, 18)$; *Middle:* Projection of digital triangle $T(ABC)$ on the functional plane. The white pixels belong to $I_{2D}\triangle A'B'C'$ but do not correspond to sides of $T(ABC)$. Dark gray pixels correspond to sides of $T(ABC)$ but do not belong to $I_{2D}\triangle(A'B'C')$. Light gray pixels are in $I_{2D}\triangle A'B'C'$ *and* correspond to sides of $T(ABC)$; *Right:* Mesh of two 3D digital triangles $T(ABC)$ and $T(ABD)$, obtained by the described algorithm. The mesh vertices are $A = (1, 8, 6)$, $B = (-8, -2, 0)$, $C(7, -8, -4)$, and $D(14, -4, -5)$.

to $\triangle ABC$. With this preparation, a *3D digital triangle* $T(ABC)$ is defined as the union of its sides $L_R(AB)$, $L_R(AC)$, and $L_R(BC)$ and the integer set $I_{3D}\triangle ABC$. Note that the discrete sides of $T(ABC)$ and the integer set of $\triangle ABC$ may contain common voxels (see Figure 16 (*Middle*).

The above constructive definition infers an algorithm for digitization of triangles and meshes of triangles. Let a mesh of a finite number of 3D triangles be given. Each triangle is specified by its three vertices that are supposed to be integer points. A triangle $\triangle ABC$ in the 3D space is then digitized as follows.

(i) Approximate the sides AB, AC, and BC by the corresponding regular 3D lines $L_R(AB)$, $L_R(AC)$, and $L_R(BC)$;
(ii) Determine the regular plane P_R^{ABC};
(iii) Find the functional plane $\pi_{P_R^{ABC}}$ of P_R^{ABC};
(iv) Find the respective projections A', B', and C' of A, B, and C on $\pi_{P_R^{ABC}}$;
(v) Determine the integer set $I_{2D}\triangle A'B'C'$ of $\triangle A'B'C'$;
(vi) Generate the integer set $I_{3D}\triangle ABC$ of $\triangle ABC$ from $I_{2D}\triangle A'B'C'$.

The union of the sides and the integer set constitutes the digital triangle $T(ABC)$. Then the triangular mesh voxelization is obtained by digitizing every triangle of the mesh. It is proved that a digital triangle generated by the above algorithm is 6-gapfree and that the obtained triangular mesh voxelization is 6-gapfree, as well. Moreover, removal of an arbitrary voxel from the obtained digital polyhedral surface causes occurrence of a 6-gap. The algorithm has linear time and space complexity in the number of the generated voxels. An example of a mesh of two digital triangles obtained through the proposed algorithm, is outlined in Figure 16 (*Right*).

8 Conclusions

Digital planarity is expected to be an even more challenging subject than digital straightness. It seems to be far from fully explored, and the authors expect further valuable contributions to this subject in near future. This article may help to focus research on important open issues such as number-theoretic characterizations or a wider collection of recognition algorithms with a more detailed comparative evaluation. Segmentations of 3D surfaces into DPS*s* will become increasingly important. Characterizations of such segmentations (e.g., "balanced in size," or "approximating convex faces"), as well as algorithms that optimize such kind of properties, are of significant interest.

Acknowledgments

End of 2003 the late Azriel Rosenfeld agreed to use parts of chapters 11 and 12 of (Klette and Rosenfeld 2004) for this reviewing article. The authors dedicate it to this outstanding researcher who established the field of digital geometry with numerous fundamental contributions.

References

1940-1959

[Morse and Hedlund 1940] M. Morse and G.A. Hedlund. Symbolic dynamics II: Sturmian sequences. *Amer. J. Math.*, 61:1–42, 1940.

1960-1969

[Freeman 1961] H. Freeman. Techniques for the digital computer analysis of chain-encoded arbitrary plane curves. In *Proc. Natl. Elect. Conf.*, **17**:421–432, 1961.

[Blankinship 1963] W.A. Blankinship A new version of the Euclidean algorithm. *American Mathematics Monthly*, 742–745, 1963.

1970-1979

[Brons 1974] R. Brons. Linguistic methods for description of a straight line on a grid. *Computer Graphics Image Processing*, 2:48–62, 1974.

1980-1989

[Kim and Rosenfeld 1982] C.E. Kim and R. Rosenfeld Convex digital solids. *IEEE Trans. Pattern Analysis Machine Int.*, 4:612–618, 1982.

[Megiddo 1983] N. Megiddo. Linear-time algorithms for linear programming in $\mathbb{R}^3$ and related problems. *SIAM Journal of Computing*, 12:759–776, 1983.

[Dorst and Smeulders 1984] L. Dorst and A. W. M Smeulders. Discrete Representation of Straight Lines. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6:450–463, 1984.

[Kim 1984] C.E. Kim. Three-dimensional digital planes. *IEEE Trans. Pattern Analysis Machine Int.*, 6:639–645, 1984.

[Megiddo 1984] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31:114–127, 1984.

[Hung 1985] S.H.Y. Hung. On the straightness of digital arcs. *IEEE Trans. Pattern Anal. Machine Intell.*, 7:203-215, 1985.

[McIlroy 1985] M. D. McIlroy. A Note on Discrete Representation of Lines. *AT&T Technical Journal*, 64(2):481–490, 1985.

[Preparata and Shamos 1985] F.P. Preparata and M.I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, 1985.

[Kaufman and Shimony 1986] A. Kaufman and E. Shimony E. 3D Scan-Conversion Algorithms for Voxel-Based Graphics. In: Proceedings of the 1986 Workshop on Interactive 3D Graphics, Chapel Hill, NC, ACM, New York, 45-75, 1986.

[Kaufman 1987] A. Kaufman. An algorithm for 3D scan-conversion of polygons. EUROGRAPHICS'87, G. Maréchal, Ed., 197–208, 1987.

[Lorensen and Cline 1987] W. E. Lorensen, H. E. Cline, Marching cubes: a high resolution 3D surface construction algorithm, in: M. C. Stone (Ed.), SIGGRAPH '87 Conference Proceedings (Anaheim, CA, July 27–31, 1987), Computer Graphics, Volume 21, Number 4, 1987, pp. 163–170.

[Forchhammer 1989] S. Forchhammer. Digital plane and grid point segments. *Computer Vision Graphics Image Processing*, **47**:373–384, 1989.

1990-1999

[Bruckstein 1991] A.M. Bruckstein. Self-similarity properties of digitized straight lines. *Contemp. Math.*, 119:1-20, 1991.

[Kim 1991] C.E. Kim and I. Stojmenović. On the recognition of digital planes in three-dimensional space. *Pattern Recognition Letters*, 12:665–669, 1991.

[Reveillès 1991] J.-P. Reveillès. Géométrie discrète, calcul en nombres entiers et algorithmique. Thèse d'état, Univ. Louis Pasteur, Strasbourg, 1991.

[Stojmenović and Tosić 1991] I. Stojmenović and R. Tosić. Digitization schemes and the recognition of digital straight lines, hyperplanes and flats in arbitrary dimensions. *Vision Geometry, Contemporary Mathematics Series*, 119:197–212, 1991.

[Lunnon and Pleasants 1992] W.F. Lunnon and P.A.B. Pleasants. Characterization of two-distance sequences. *J. Australian Mathematical Society, Ser. A*, **53**:198–218, 1992.

[Lindenbaum and Bruckstein 1993] M. Lindenbaum and A. M. Bruckstein. On Recursive, O(n) Partitioning of a Digitized Curve into Digital Straight Segments, *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 15(9):949–953, 1993.

[Veelaert 1993] P. Veelaert. On the flatness of digital hyperplanes. *J. Math. Imaging and Vision*, 3:205–221, 1993.

[Voss 1993] K. Voss. *Discrete Images, Objects, and Functions in* $\mathbb{Z}^n$. Springer, Berlin, 1993.

[Debled-Renesson and Reveillès 1994] I. Debled-Rennesson and J.-P. Reveillès. A new approach to digital planes. *Vision Geometry III*, SPIE-2356:12–21, 1994.

[Ito and Ohtsuki al. 1994] S. Ito and M. Ohtsuki. Parallelogram tilings and Jacobi-Perron algorithm. *Tokyo J. Math* **17** 33–58, 1994.

[Veelaert 1994] P. Veelaert. Digital planarity of rectangular surface segments. *IEEE Pattern Analysis and Machine Int*, 16:647–652, 1994.

[Debled-Rennesson 1995] I. Debled-Rennesson. Etude et reconnaissance des droites et plans discrets. *PhD Thesis*, Université Louis Pasteur, Strasbourg, France, 1995.

[Figueiredo and Reveillès 1995] O. Figueiredo and J.-P. Reveillès. A contribution to 3D digital lines. *5th Int. Workshop Discrete Geometry for Computer Imagery*, 187–198, Clermont-Ferrand, France, 1995.

[Françon 1995] J. Françon. Arithmetic planes and combinatorial manifolds. *5th Int. Workshop Discrete Geometry for Computer Imagery*, 209–217, Clermont-Ferrand, France, 1995.

[Reveillès 1995] J.-P. Reveillès. Combinatorial pieces in digital lines and planes. *Vision Geometry IV*, Vol. 2573, SPIE, 23–34, 1995.

[Fukuda and Prodon 1996] K. Fukuda and A. Prodon. Double description method revisited. *Combinatorics and Computer Science*, Springer, LNCS-1120:91–111, 1996.

[Françon et al. 1996] J. Françon, J.M. Schramm and M. Tajine. Recognizing arithmetic straight lines and planes. 6th Int. Conf. *Discrete Geometry for Computer Imagery*, Springer, LNCS-1176:141–150, 1996.

[Klette et al. 1996] R. Klette, I. Stojmenović, and J. Žunić. A parametrization of digital planes by least square fits and generalizations. *Graphical Models Image Processing*, **58**:295–300, 1996.

[Andres et al. 1997] E. Andres, R. Acharya, and C. Sibata. Discrete analytical hyperplanes. *Graphical Models and Image Processing*, 59:302–309, 1997.

[Andres et al. 1997a] E. Andres, Ph. Nehlig and J. Françon. Supercover of straight lines, planes and triangles. 7th Int. Conf. *Discrete Geometry for Computer Imagery*, LNCS-1347:87–98, 1997.

[Andres et al. 1997b] E. Andres, Ph. Nehlig and J. Françon. Tunnel-free supercover 3D polygons and polyhedra. EUROGRAPHICS'97, D. Fellner and L. Szirmay-Kalos (Guest Eds.) C3-C13, 1997.

[Nivat 1997] M. Nivat. Invited talk at ICALP, Bologna 1997.

[Schramm 1997] J.-M. Schramm. Coplanar Tricubes. 7th Int. Conf. *Discrete Geometry for Computer Imagery*, LNCS-1347:87–98, 1997.

[Vittone and Chassery 1997] J. Vittone and J.-M. Chassery. Coexistence of Tricubes in Digital Naive Plane. 7th Int. Conf. *Discrete Geometry for Computer Imagery*, LNCS-1347:99–110, 1997.

[Vuillon 1998] L. Vuillon. Combinatoire des motif d'une suite sturmienne bidimensionnelle. *Theoretical Computer Science* **209** 261–285, 1998.

[Cassaigne 1999] J. Cassaigne. Two-dimensional sequences of complexity $mn + 1$. *J. Automatic Language Combinatorics* **4** 153–170, 1999.

[Françon and Papier 1999] J. Françon and L. Papier. Polyhedrization of the boundary of a voxel object. 8th Int. Workshop *Discrete Geometry for Computer Imagery*, Springer, LNCS-1568:425–434, 1999.

[Gérard 1999] Y. Gérard. Local configurations of digital hyperplanes. 8th Int. Workshop *Discrete Geometry for Computer Imagery*, Springer, LNCS-1568:65–75, 1999.

[Vittone and Chassery 1999] J. Vittone and J.-M. Chassery. (n, m)-cubes and Farey nets for naive planes understanding. 8th Int. Workshop *Discrete Geometry for Computer Imagery*, Springer, LNCS-1568:76–87, 1999.

2000-...

[Berthé and Vuillon 2000a] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of Sturmian words. *Discrete Mathematics* **223** 27–53, 2000.

[Berthé and Vuillon 2000b] V. Berthé and L. Vuillon. Suites doubles de basse complexité. *J. Théor. Nombres Bordeaux* **12** 179–208, 2000.

[Barneva et al. 2000] R.P. Barneva, V.E. Brimkov, and Ph. Nehlig. Thin discrete triangular meshes. Theoretical Computer Science, Elsevier, **246** (1-2) pages 73-105, 2000.

[Brimkov et al. 2000a] V.E. Brimkov, E. Andres, and R.P. Barneva. Object discretizations in higher dimensions. In G. Borgefors, I. Nystrom, and G. Sanniti di Baja, editors, *Discrete Geometry for Computer Imagery*, pages 210–221. Springer, Berlin, 2000.

[Brimkov et al. 2000b] V.E. Brimkov, R.P. Barneva and Ph. Nehlig. Minimally thin discrete triangulations. In: *Volume Graphics*, A. Kaufman, R. Yagel, M. Chen (Eds.), Springer Verlag, Chapter 3, pages 51-70, 2000.

[Lachaud and Montanvert 2000] J.-O. Lachaud, A. Montanvert, Continuous analogs of digital boundaries: A topological approach to iso-surfaces, Graphical Models and Image Processing 62 (2000) 129–164.

[Vittone and Chassery 2000] J. Vittone and J.-M. Chassery. Recognition of digital naive planes and polyhedrization. 9th Int. Workshop *Discrete Geometry for Computer Imagery*, Springer, LNCS-1953:296–307, 2000.

[Arnoux et al. 2001] P. Arnoux, V. Berthé, H. Ei and S. Ito. Tilings, quasicrystals, discrete planes, generalized substitutions, and multidimensional continued fractions. *Discrete Mathematics and Theoretical Computer Science Proceedings* **AA (DM-CCG)** 059–078, 2001.

[Berthé and Vuillon 2001] V. Berthé and L. Vuillon, Palindromes and two-dimensional Sturmian sequences. *J. Autom. Lang. Comb.* **6** 121–138, 2001.

[Klette and Sun 2001] R. Klette and H.-J. Sun. Digital planar segment based polyhedrization for surface area estimation. In C. Arcelli, L.P. Cordella, and G. Sanniti di Baja, editors, *Visual Form 2001*, pages 356–366. Springer, Berlin, 2001.

[Rosenfeld and Klette 2001] A. Rosenfeld and R. Klette. Digital straightness. In *Electronic Notes in Theoretical Computer Science* **46**, 2001.

[Berthé and Tijdeman 2002] V. Berthé and R. Tijdeman. Balance properties of multi-dimensional words. *Theoretical Computer Science* **273** 197–224, 2002.

[Brimkov 2002] V.E. Brimkov. Digital flatness and related combinatorial problems. CITR-TR-120, Computer Science Department, University of Auckland, 2002. To appear in *Discrete Applied Mathematics*, Elsevier, 2004.

[Brimkov et al. 2002] V.E. Brimkov, E. Andres, and R.P. Barneva. Object discretizations in higher dimensions. *Pattern Recognition Letters*, **23**:623–636, 2002.

[Brimkov and Barneva 2002] V.E. Brimkov and R.P. Barneva. Graceful planes and lines. *Theoretical Computer Science*, **283**:151–170, 2002.

[Buzer 2002] L. Buzer. An incremental linear time algorithm for digital line and plane recognition using a linear incremental feasibility problem. 10th Int. Conf. *Discrete Geometry for Computer Imagery*, Springer, LNCS-2301:372–381, 2002.

[Lothaire 2002] M. Lothaire (ed.). *Algebraic Combinatorics on Words.* Cambridge University Press, Cambridge, 2002.

[Andres 2003] E. Andres. Discrete linear objects in dimension n: the standard model. *Graphical Models* **65** 92–111, 2003.

[Brimkov and Barneva 2003] V.E. Brimkov and R.P. Barneva. Connectivity of discrete planes. CITR-TR-125, Computer Science Department, University of Auckland, 2003. To appear in *Theoretical Computer Science*, Elsevier, 2004.

[Coeurjolly et al. 2003] D. Coeurjolly, I. Sivignon, F. Dupont, F. Feschet and J.-M. Chassery. Digital plane preimage structure. *Int. Workshop on Combinatorial Image Analysis 2003*, Elsevier ENDM, 2003.

[Epifanio et al. 2003] Ch. Epifanio, M. Koskas and F. Mignosi. On a conjecture on bidimensional words. *Theoretical Computer Science* **299**(1-3) 123–150, 2003.

[Coeurjolly et al. 2004] D.. Coeurjolly, A. Guillaume and I. Sivignon. Reversible discrete volume polyhedrization using Marching Cubes simplification. *Vision Geometry XII*, Vol. 5300, SPIE, 2004.

[Klette and Rosenfeld 2004] R. Klette and A. Rosenfeld. Digital Geometry - Geometric Methods for Digital Picture Analysis. Morgan Kaufmann, San Francisco, 2004.